Combinatorial Properties and Recognition of Unit Square Visibility Graphs[1]

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  2-dim. Euclidean space, 3-dim. Euclidean space, ...

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Main motivation:
Graphs that model real-world systems are often geometric graphs
e. g.: radio transmitters (2d/3d space, proximity as edge relation)

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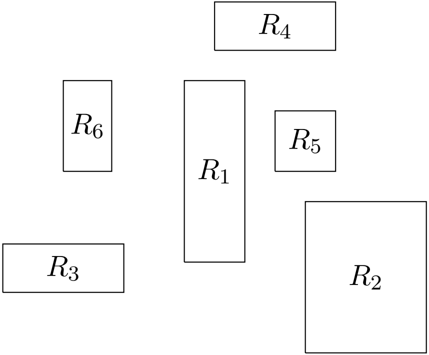

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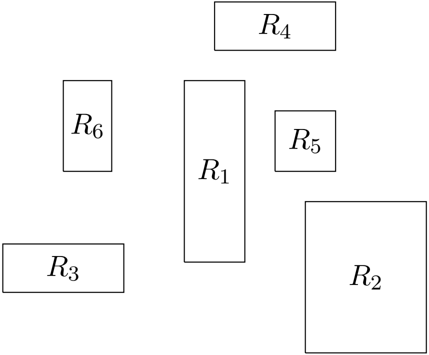

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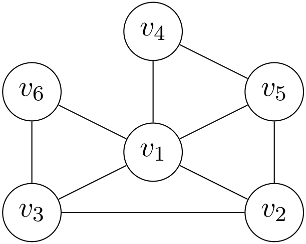

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USV
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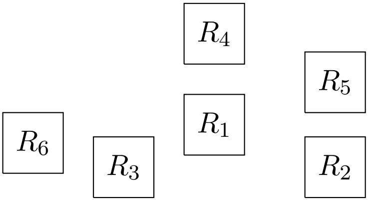

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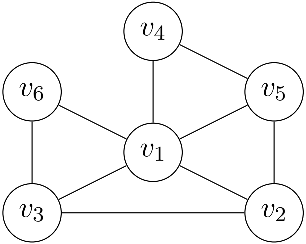

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  - only planar graphs
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Additional motivation from graph drawing:

- good readability properties:
  - only rectangular edge crossings,
  - angles between adjacent edges are rectangular,

Grid Case

Unit square grid visibility graphs (USGV, USGV_w):
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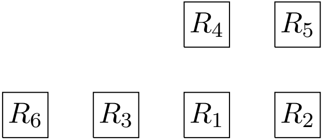

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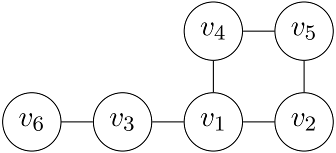

Research Questions

- Combinatorial properties: what kind of graphs can be represented by unit square layouts?

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- Recognition problem: decide whether a given graph can be represented by a unit square layout.

Unit Square Grid Visibility Graphs
(USGV)

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- USGV $=$ USGV$_w$.
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Rectilinear Drawing (RLG)

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(edges do not intersect non-adjacent vertices).

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- USGV = RLG (all combinatorial results of RLG apply to USGV).
  USGV $\subseteq$ RLG: an USGV-layout is a RLG-drawing
  USGV $\supseteq$ RLG: vertex-points $\rightarrow$ squares, delete unwanted edges

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- USGV do not contain $K_{1,5}, K_{2,3}$ or K_3 as subgraphs
- USGV contains non-bipartite graphs (e. g., C_5)

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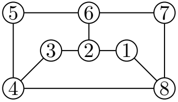

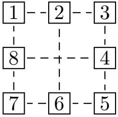

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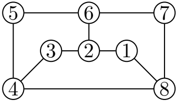
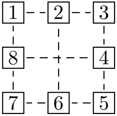
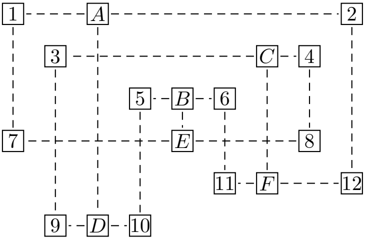
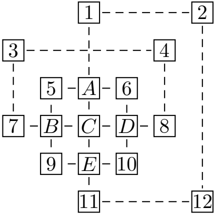




- USGV contains non-planar graphs (subdivisions of $K_{3,3}$ and K_5):





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- Characterisations of cycles, complete graphs, complete bipartite graphs and trees within USGV:
  - $C_i \in \mathsf{USGV} \iff i \geq 4$,
  - $K_i \in \mathsf{USGV} \iff i \leq 2$,
  - $K_{i,j} \in \mathsf{USGV}$, $i \leq j \iff (i = 1 \text{ and } j \leq 4) \text{ or } (i = 2 \text{ and } j = 2)$.
  - Tree $T \in \mathsf{USGV} \iff T$ has maximum degree ≤ 4.

USGV – Recognition Problem, Known Results

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LRDU-Restricted variant:
Given graph G and $R : E \to \{\mathsf{L}, \mathsf{R}, \mathsf{D}, \mathsf{U}\}$ (the LRDU-Restriction), does there exist a layout representing G with
$R(\{u, v\}) = \mathsf{R} \Rightarrow$ unit square for v is to the right of unit square for u,
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Can be solved for RLG in $\mathrm{O}(|E| \cdot |V|)$ (so also for USGV).

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Given graph G and $w, h \in \mathbb{N}$, does there exist a layout representing G with height h and width w?

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Reduction does not work for USGV: transforming RLG drawing into USGV layout requires more space.

USGV – Recognition Problem

3-Partition

Input: $B \in \mathbb{N}$, multiset $A = \{a_1, a_2, \ldots, a_{3m}\} \subseteq \mathbb{N}$ with $\frac{B}{4} < a_i < \frac{B}{2}$ and $\sum_{i=1}^{3m} a_i = mB$.
Question: $\exists$ partition $A = A_1, \ldots, A_m$ with $\sum_{a \in A_j} = B$, $1 \leq j \leq m$?

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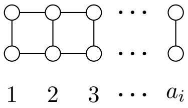

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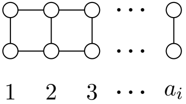

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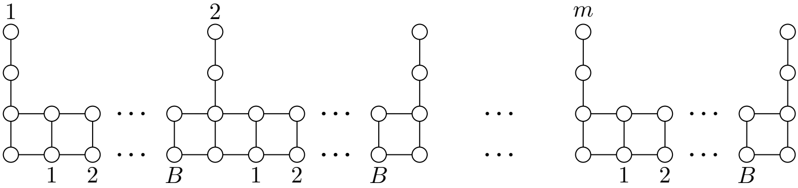

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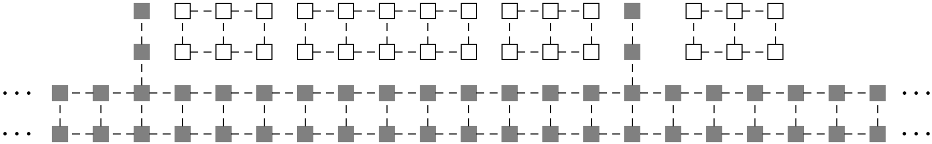
$a_{i_1} = 3$, $a_{i_2} = 5$, $a_{i_3} = 3$ and $B = 11$



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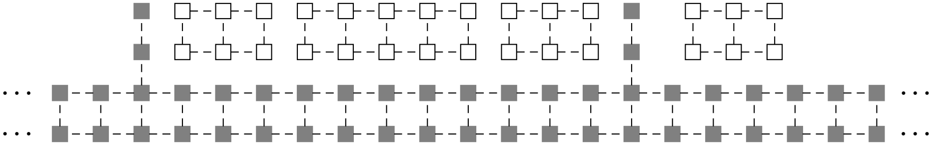

G has a *weak* $(7 \times (2(mB + m + 1) - 1))$ unit square grid layout $\iff$
$\exists$ partition $A = A_1, \ldots, A_m$ with $\sum_{a \in A_j} = B$, $1 \leq j \leq m$.

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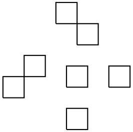
Area minimisation variant of LRDU-restricted Recognition Problem for USGV still open!

Unit Square Visibility Graphs
(USV)

USV – Combinatorial Results

Some Examples:



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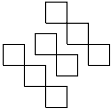
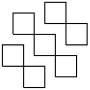


$K_{2,6}$



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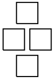
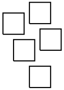


K_4



K_5 with one missing edge

USV – Combinatorial Results

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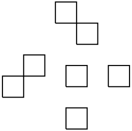

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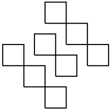

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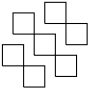

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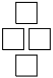

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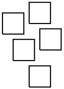

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Some simple observations:

- Every graph with at most 4 vertices is in USV.

USV – Combinatorial Results

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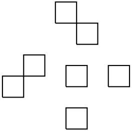

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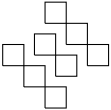

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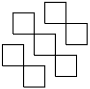

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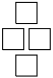

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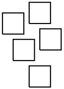

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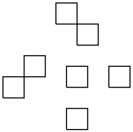
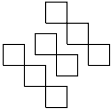
- The degree of graphs in USV is not bounded ()

USV – Combinatorial Results

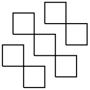
Some Examples:



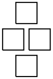
$K_{1,6}$



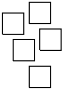
$K_{2,6}$



$K_{3,4}$



K_4



K_5 with one missing edge

Some simple observations:

- Every graph with at most 4 vertices is in USV.
- The degree of graphs in USV is not bounded ()
- Vertex has degree $\geq 7 \Rightarrow \exists$ paths between some of its neighbours. (However, $K_{1,n}$ may exist as induced subgraph for every n.)

USV – Combinatorial Results

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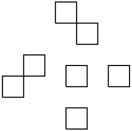

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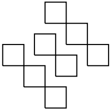

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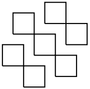

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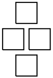

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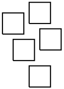

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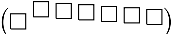
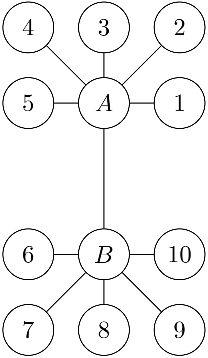
- The degree of graphs in USV is not bounded ()
- Vertex has degree $\geq 7 \Rightarrow \exists$ paths between some of its neighbours. (However, $K_{1,n}$ may exist as induced subgraph for every n.)
- USV $\subsetneq$ USV$_w$ (seperated, e. g., by $K_{1,7}$)

USV – Combinatorial Results



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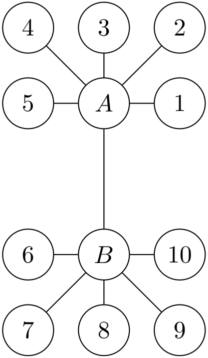

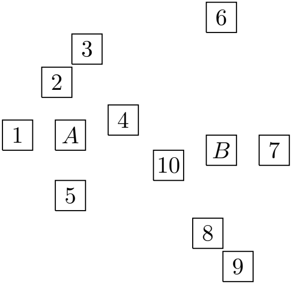

USV - Recognition Problem

- Recognition problem for USV is in NP.
  - If $G \in \mathsf{USV}$, then there is a $n \times n$ layout.
  - No arbitrary small "shifting" between unit squares necessary.

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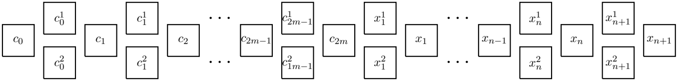
- Recognition problem for USV is NP-hard.
  - Reduction from NAE-3SAT (not-all-equal 3-satisfiability).
  - Sketch follows ...

USV - Recognition Problem, Sketch of Reduction

backbone:



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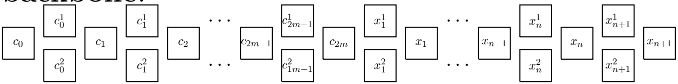
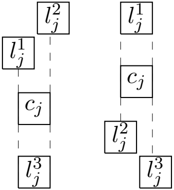
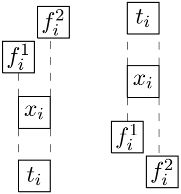


clause gadgets:



variable gadgets:



USV - Recognition Problem, Sketch of Reduction

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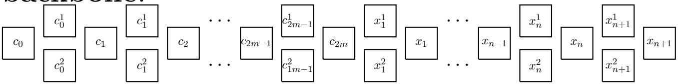
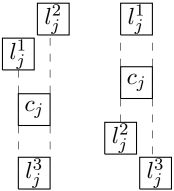
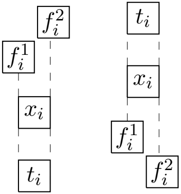


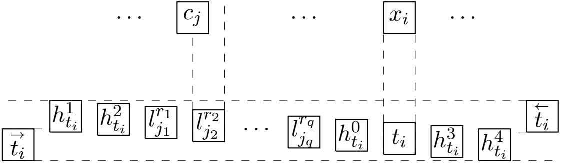
clause gadgets:



variable gadgets:



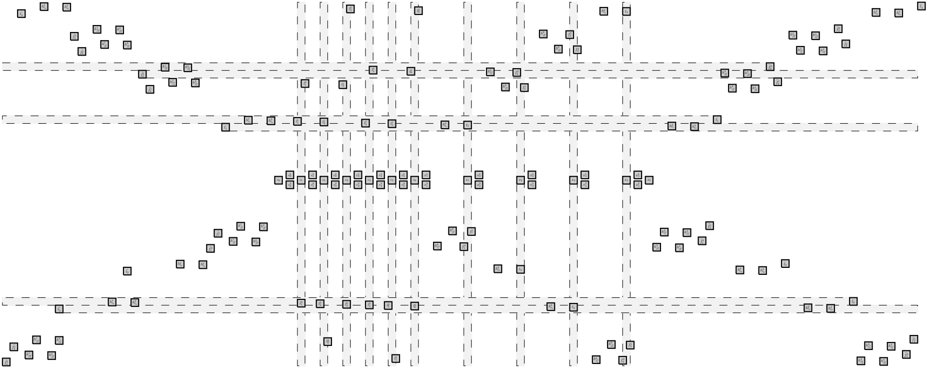
variable paths:



USV - Recognition Problem, Full Reduction

Formula:

$\{c_1, c_2, c_3\}$ with $c_1 = \{x_1, \bar{x}_2, x_3\}, c_2 = \{x_1, x_3, \bar{x}_4\}, c_3 = \{\bar{x}_2, x_3, x_4\}$:



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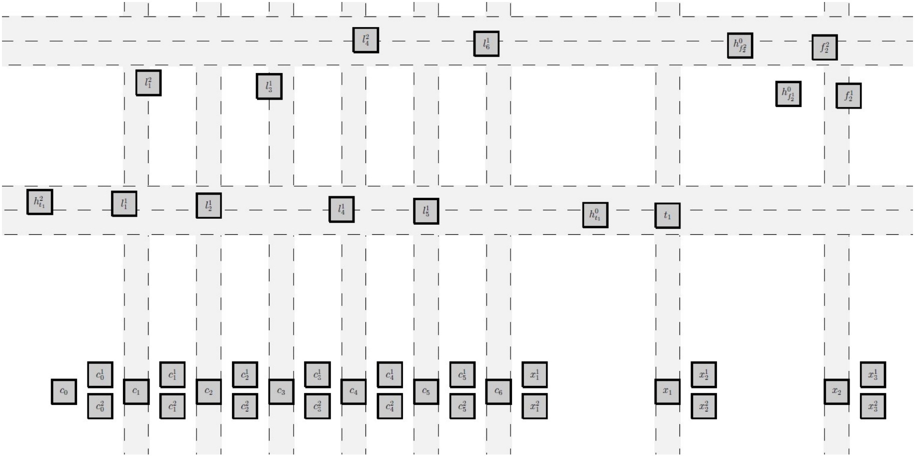

Further Research

- Area minimisation variant of LRDU-restricted Recognition Problem for USGV.

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- Recognition Problem for USV_w.

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- Practically more realistic: $\{\frac{\ell}{k} \mid \ell \in \mathbb{N}\}^2$ grid (with k treated as parameter).

Thank you very much for your attention