Closure Properties of Pattern Languages

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Basic Definitions and Notation

\[ \Sigma \quad \text{Terminal} \quad \{a, b, c\} \]
### Basic Definitions and Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Terminals</th>
<th>Variables</th>
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<tbody>
<tr>
<td>$\Sigma$</td>
<td>{a, b, c}</td>
<td>{x_1, x_2, x_3, \ldots}</td>
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Basic Definitions and Notation

$\Sigma$ \hspace{1cm} Terminals \hspace{1cm} \{a, b, c\}

$X$ \hspace{1cm} Variables \hspace{1cm} \{x_1, x_2, x_3, \ldots\}

$w \in \Sigma^*$ \hspace{1cm} Word \hspace{1cm} abaacba
Basic Definitions and Notation

\[ \Sigma \quad \text{Terminals} \quad \{a, b, c\} \]

\[ X \quad \text{Variables} \quad \{x_1, x_2, x_3, \ldots\} \]

\[ w \in \Sigma^* \quad \text{Word} \quad \text{abaacba} \]

\[ \alpha \in (\Sigma \cup X)^+ \quad \text{Pattern} \quad \alpha := x_1 ax_2 x_1 bax_2 x_1 x_3 \]
Pattern Languages

Morphism Mapping $h : \Gamma_1^* \rightarrow \Gamma_2^*$ with $h(x \cdot y) = h(x) \cdot h(y)$; $h$ is nonerasing iff, for every $a \in \Gamma_1$, $h(a) \neq \varepsilon$. 
Pattern Languages

**Morphism** Mapping \( h : \Gamma_1^* \to \Gamma_2^* \) with \( h(x \cdot y) = h(x) \cdot h(y) \); \( h \) is nonerasing iff, for every \( a \in \Gamma_1 \), \( h(a) \neq \varepsilon \).

**Substitution** Morphism \( h : (\Sigma \cup X)^* \to \Sigma^* \) with \( h(a) = a, \ a \in \Sigma \).
Pattern Languages

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E-pattern lang.  $L_{E,\Sigma}(\alpha) := \{h(\alpha) \mid h \text{ is a substitution}\}$.
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E-pattern lang.  \( L_{E,\Sigma}(\alpha) := \{ h(\alpha) \mid h \) is a substitution \}\).

NE-pattern lang.  \( L_{NE,\Sigma}(\alpha) := \{ h(\alpha) \mid h \) is nonerasing substitution \}\).
An Example

\[ \alpha = x_1 \text{ aa } x_2 \text{ } x_1 \text{ } x_2 \text{ } cb \text{ } x_1 \]
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\[ \alpha = x_1 \text{ aa } x_2 \text{ x_1 } x_2 \text{ cb } x_1 \]

\[ \text{acaaabcbaacabcbacbac} \]
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\[ ac a a a b c b a c a b c b a c b a c b \]
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\[ \text{acaaabcbaacabcbacbac} \]

\[ h(\alpha) = \text{acaaabcbaacabcbacbac} \in L_{NE, \{a,b,c\}}(\alpha), \]
where \( h(x_1) = \text{ac} \), \( h(x_2) = \text{abcba} \), \( h(a) = a \), \( h(b) = b \).
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\[ \text{acaaabcbaacabcbacbac} \]

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where \( h(x_1) = \text{ac}, \ h(x_2) = \text{abcba}, \ (h(a) = a, \ h(b) = b). \)

\[ \text{ccbaaccbcbccb} \notin L_{\text{NE,\{a,b,c\}}}(\alpha) \]
An Example

\[ \alpha = x_1 \text{ aa } x_2 x_1 x_2 \text{ cb } x_1 \]

\[ acaaabcbaacabcbacbac \]

\[ h(\alpha) = acaaabcbaacabcbacbac \in L_{NE,\{a,b,c\}}(\alpha), \]
where \( h(x_1) = ac, \ h(x_2) = abcba, \ (h(a) = a, \ h(b) = b). \]

\[ ccbaaccbcbccb \notin L_{NE,\{a,b,c\}}(\alpha) \]
\[ ccbaaccbcbccb \in L_{E,\{a,b,c\}}(\alpha) \]
Some Background Information on Pattern Languages

- Introduced by Angluin in 1979. (Original motivation: inductive inference)

Later investigated from a purely language theoretical point of view. Independently developed in the pattern matching community. Practically applied in so-called regular expressions with backreferences (Perl, Java, Python, ...). Relations to combinatorics on words: pattern avoidability, ambiguity of morphisms, word equations, equality sets.
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- Equivalence (NE-case) is trivial.
- Equivalence (E-case) is open.
- Inclusion (terminal-free, E-case) is NP-complete.
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Closure Properties

Angluin 1979:

Pattern Languages are not closed under

- union
  \[ L_{NE,\Sigma}(a) \cup L_{NE,\Sigma}(b) = \{a, b\} \]

- intersection
  \[ L_{NE,\Sigma}(a) \cap L_{NE,\Sigma}(b) = \emptyset \]

- complement
  \[ \{a, b\}^* \setminus L_{NE,\Sigma}(a) \]

- Kleene plus
  \[ (L_{NE,\{a,b\}}(a))^* \ (L_{NE,\{a,b\}}(a))^+ \]

- homomorphism
  \[ h(L_{NE,\{a,b\}}(x)) = (L(a))^+, \ h(a) = h(b) = a \]

- inv. homo.
  \[ g^{-1}(L_{NE,\{a,b\}}(aaa)) = \{aaa, ab, ba\}, \ g(a) = a, \ g(b) \]
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- inv. homo.
  \[ g^{-1}(L_{\text{NE},\{a,b\}}(aaa)) = \{aaa, ab, ba\}, \ g(a) = a, \ g(b) \]

Pattern Languages are **closed** under

- concatenation
  \[ L(\alpha) \cdot L(\beta) = L(\alpha \cdot \beta) \]
- reversal
  \[ (L(\alpha))^R = L(\alpha^R) \]
Motivation for Investigating Closure Properties

- One of the most classical and fundamental question in language theory.
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- One of the most classical and fundamental question in language theory.
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- In the case of pattern languages the existing closure properties fail to contribute to our understanding of their intrinsic properties.
- All examples for non-closure require terminal symbols in the patterns (what about the closure of terminal-free pattern languages).
- Can we characterise those pairs \((\alpha, \beta)\) of patterns, for which \(L(\alpha) \cup L(\beta)\) or \(L(\alpha) \cap L(\beta)\) are pattern languages?
Canonical Way of Expressing (NE/E)-pattern languages by unions of (E/NE)-pattern languages
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- Every E-pattern language is the finite union of NE-pattern languages.
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Let $\Sigma = \{a, b\}$ and $\alpha = x_1x_2x_2x_1x_3x_1$.
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Let $\Sigma = \{a, b\}$ and $\alpha = x_1 x_2 x_2 x_1 x_3 x_1$.

\[
\begin{align*}
\beta_1 &= x_1 x_2 x_2 x_1 x_3 x_1, \\
\beta_2 &= x_2 x_2 x_3, \\
\beta_3 &= x_1 x_1 x_3 x_1, \\
\beta_4 &= x_1 x_2 x_2 x_1 x_1, \\
\beta_5 &= x_3, \\
\beta_6 &= x_2 x_2, \\
\beta_7 &= x_1 x_1 x_1.
\end{align*}
\]

$L_{E,\Sigma}(\alpha) = \bigcup_{i=1}^{6} L_{NE,\Sigma}(\beta_i)$. 

Is this the only way of how unions of E- or unions of NE-pattern languages can be a NE- or a E-pattern languages, respectively?
Canonical Way of Expressing (NE/E)-pattern languages by unions of (E/NE)-pattern languages

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Let $\Sigma = \{a, b\}$ and $\alpha = x_1 x_2 x_2 x_1 x_3 x_1$.

$\beta_1 = x_1 x_2 x_2 x_1 x_3 x_1$,  $\gamma_1 = a x_1 a x_2 a x_2 a x_1 a x_3 a x_1$,  
$\beta_2 = x_2 x_2 x_3$,  $\gamma_2 = b x_1 a x_2 a x_2 b x_1 a x_3 b x_1$,  
$\beta_3 = x_1 x_1 x_3 x_1$,  $\gamma_3 = a x_1 b x_2 b x_2 a x_1 a x_3 a x_1$,  
$\beta_4 = x_1 x_2 x_2 x_1 x_1$,  $\gamma_4 = a x_1 a x_2 a x_2 a x_1 b x_3 a x_1$,  
$\beta_5 = x_3$,  $\gamma_5 = a x_1 b x_2 b x_2 a x_1 b x_3 a x_1$,  
$\beta_6 = x_2 x_2$,  $\gamma_6 = b x_1 a x_2 a x_2 b x_1 b x_3 b x_1$,  
$\beta_7 = x_1 x_1 x_1$.  $\gamma_7 = b x_1 b x_2 b x_2 b x_1 a x_3 b x_1$,  

$L_{E, \Sigma}(\alpha) = \bigcup_{i=1}^{6} L_{NE, \Sigma}(\beta_i)$.  

$L_{NE, \Sigma}(\alpha) = \bigcup_{i=1}^{8} L_{E, \Sigma}(\gamma_i)$.  

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Let $\Sigma = \{a, b\}$ and $\alpha = x_1x_2x_2x_1x_3x_1$.

$\beta_1 = x_1x_2x_2x_1x_3x_1$, $\gamma_1 = ax_1ax_2ax_2ax_1ax_3ax_1$,
$\beta_2 = x_2x_2x_3$, $\gamma_2 = bx_1ax_2ax_2bx_1ax_3bx_1$,
$\beta_3 = x_1x_1x_3x_1$, $\gamma_3 = ax_1bx_2bx_2ax_1ax_3ax_1$,
$\beta_4 = x_1x_2x_2x_1x_1$, $\gamma_4 = ax_1ax_2ax_2ax_1bx_3ax_1$,
$\beta_5 = x_3$, $\gamma_5 = ax_1bx_2bx_2ax_1bx_3ax_1$,
$\beta_6 = x_2x_2$, $\gamma_6 = bx_1ax_2ax_2bx_1bx_3bx_1$,
$\beta_7 = x_1x_1x_1$, $\gamma_7 = bx_1bx_2bx_2bx_1ax_3bx_1$.

$L_{E,\Sigma}(\alpha) = \bigcup_{i=1}^{6} L_{NE,\Sigma}(\beta_i)$. $L_{NE,\Sigma}(\alpha) = \bigcup_{i=1}^{8} L_{E,\Sigma}(\gamma_i)$.

Is this the only way of how unions of E- or unions of NE- pattern languages can be a NE- or a E-pattern languages, respectively?
Closure of Terminal-Free Pattern Languages

Terminal-free pattern languages . . .

- . . . have been a recent focus of interest in the research of pattern languages.
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- . . . have better decidability properties (inclusion and equivalence is decidable in the E-case).
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- . . . have better decidability properties (inclusion and equivalence is decidable in the E-case).

- . . . have open closure properties.
Union of Terminal-Free Pattern Languages

**Theorem**

Let \( Z, Z' \in \{ E, NE \} \) and \( \alpha, \beta, \gamma \) patterns.

\[
L_{Z, \Sigma}(\alpha) \cup L_{Z, \Sigma}(\beta) = L_{Z', \Sigma}(\gamma) \\
\iff \\
L_{Z, \Sigma}(\alpha) \subseteq L_{Z, \Sigma}(\beta) \text{ and } L_{Z, \Sigma}(\beta) = L_{Z', \Sigma}(\gamma) \text{ or } \text{ or } \\
L_{Z, \Sigma}(\beta) \subseteq L_{Z, \Sigma}(\alpha) \text{ and } L_{Z, \Sigma}(\alpha) = L_{Z', \Sigma}(\gamma).
\]
Theorem

Let $Z, Z' \in \{E, NE\}$ and $\alpha, \beta, \gamma$ patterns.

\[ L_{Z,\Sigma}(\alpha) \cup L_{Z,\Sigma}(\beta) = L_{Z',\Sigma}(\gamma) \]

\[ \iff \]

\[ L_{Z,\Sigma}(\alpha) \subseteq L_{Z,\Sigma}(\beta) \text{ and } L_{Z,\Sigma}(\beta) = L_{Z',\Sigma}(\gamma) \text{ or } \]

\[ L_{Z,\Sigma}(\beta) \subseteq L_{Z,\Sigma}(\alpha) \text{ and } L_{Z,\Sigma}(\alpha) = L_{Z',\Sigma}(\gamma). \]

\[ \Rightarrow \text{ full characterisation of } L_z(\alpha) \cup L_z(\beta) = L_{Z'}(\gamma), \ Z, Z' \in \{E, NE\}. \]
Union of Terminal-Free Pattern Languages

**Theorem**

Let $Z, Z' \in \{E, NE\}$ and $\alpha, \beta, \gamma$ patterns.

$$L_{Z, \Sigma(\alpha)} \cup L_{Z, \Sigma(\beta)} = L_{Z', \Sigma(\gamma)}$$

$\iff$

$$L_{Z, \Sigma(\alpha)} \subseteq L_{Z, \Sigma(\beta)} \text{ and } L_{Z, \Sigma(\beta)} = L_{Z', \Sigma(\gamma)} \text{ or }$$

$$L_{Z, \Sigma(\beta)} \subseteq L_{Z, \Sigma(\alpha)} \text{ and } L_{Z, \Sigma(\alpha)} = L_{Z', \Sigma(\gamma)}.$$

$\Rightarrow$ full characterisation of $L_Z(\alpha) \cup L_Z(\beta) = L_{Z'}(\gamma), Z, Z' \in \{E, NE\}$.

Inclusion is decidable for terminal-free E-pattern languages, but still open for terminal-free NE-pattern languages.
Intersection of Terminal-Free Pattern Languages

Theorem

Let $Z \in \{E, NE\}$. Then $L_{Z,\Sigma}(x_1x_1) \cap L_{Z,\Sigma}(x_1x_1x_1) = L_{Z,\Sigma}(x_1^6)$. 
Intersection of Terminal-Free Pattern Languages

**Theorem**

Let $Z \in \{E, NE\}$. Then $L_{Z, \Sigma}(x_1x_1) \cap L_{Z, \Sigma}(x_1x_1x_1) = L_{Z, \Sigma}(x_1^6)$.

**Theorem**

$L_{NE, \Sigma}(x_1x_2x_1) \cap L_{NE, \Sigma}(x_1x_1x_2)$ is not a terminal-free NE-pattern language.
Intersection of Terminal-Free Pattern Languages

**Theorem**

Let $Z \in \{E, NE\}$. Then $L_{Z,\Sigma}(x_1 x_1) \cap L_{Z,\Sigma}(x_1 x_1 x_1) = L_{Z,\Sigma}(x_1^6)$.

**Theorem**

$L_{NE,\Sigma}(x_1 x_2 x_1) \cap L_{NE,\Sigma}(x_1 x_1 x_2)$ is not a terminal-free NE-pattern language.

**Theorem**

$L_{E,\Sigma}(x_1 x_2 x_1^2 x_2 x_1^3 x_2^2) \cap L_{E,\Sigma}(x_3 x_4^2 x_3^2 x_4^6 x_3^3)$ is not a tf-E-pattern language.
Proof Sketch

Let $\alpha = x_1 x_2 x_1^2 x_2 x_1^3 x_2^2$ and $\beta = x_3 x_4^2 x_3^2 x_4^6 x_3^3$. 
Proof Sketch

Let $\alpha = x_1 x_2 x_1^2 x_2 x_1^3 x_2^2$ and $\beta = x_3 x_4^2 x_3^2 x_4^6 x_3^3$.

$L_{E,\Sigma}(\alpha) \cap L_{E,\Sigma}(\beta)$ equals the solutions of

$$x_1 x_2 x_1 x_2 x_1 x_1 x_1 x_2 x_2 = x_3 x_4 x_4 x_3 x_3 x_4 x_4 x_4 x_4 x_4 x_4 x_4 x_4 x_3 x_3 x_3 .$$
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Let $\alpha = x_1 x_2 x_1^2 x_2 x_1^3 x_2^2$ and $\beta = x_3 x_4^2 x_3^2 x_4^6 x_3^3$.

$L_{E,\Sigma}(\alpha) \cap L_{E,\Sigma}(\beta)$ equals the solutions of

$$x_1 x_2 x_1 x_1 x_2 \quad x_1 x_1 x_1 x_2 x_2 = x_3 x_4 x_4 x_3 x_3 x_4 x_4 \quad x_4 x_4 x_4 x_4 x_3 x_3 x_3 x_3.$$
Proof Sketch

Let \( \alpha = x_1 x_2 x_1^2 x_2 x_1^3 x_2^2 \) and \( \beta = x_3 x_4^2 x_3^2 x_4^6 x_3^3 \).

\( L_{E,\Sigma}(\alpha) \cap L_{E,\Sigma}(\beta) \) equals the solutions of

\[
\begin{align*}
x_1 x_2 x_1 x_1 x_2 &= x_3 x_4 x_4 x_3 x_3 x_4 x_4 \\
x_1 x_1 x_1 x_2 x_2 &= x_4 x_4 x_4 x_4 x_3 x_3 x_3
\end{align*}
\]
Proof Sketch

Let $\alpha = x_1 x_2 x_1^2 x_2 x_1^3 x_2^2$ and $\beta = x_3 x_4^2 x_3^2 x_4^6 x_3^3$.

$L_{E,\Sigma}(\alpha) \cap L_{E,\Sigma}(\beta)$ equals the solutions of

\[
\begin{align*}
x_1 x_2 x_1 x_1 x_2 &= x_3 x_5 x_3 x_3 x_5 \\
x_1 x_1 x_1 x_2 x_2 &= x_5 x_5 x_3 x_3 x_3 \\
x_5 &= x_4 x_4
\end{align*}
\]
Proof Sketch

Let \( \alpha = x_1 x_2 x_1^2 x_2 x_1^3 x_2^2 \) and \( \beta = x_3 x_4^2 x_3^2 x_4^6 x_3^3 \).

\( L_{E,\Sigma}(\alpha) \cap L_{E,\Sigma}(\beta) \) equals the solutions of

\[
\begin{align*}
x_1 x_2 x_1 x_1 x_2 &= x_3 x_5 x_3 x_3 x_5 \\
x_1 x_1 x_1 x_2 x_2 &= x_5 x_5 x_3 x_3 x_3 \\
x_5 &= x_4 x_4
\end{align*}
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$\Rightarrow$ all solutions to the equations are periodic.
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$\Rightarrow$ all solutions to the equations are periodic.

Lemma: If $\alpha = \beta$ has only periodic solutions and $L_{E,\Sigma}(\alpha) \cap L_{E,\Sigma}(\beta)$ is a terminal-free E-pattern language, then $a^k \in L_{E,\Sigma}(\alpha) \cap L_{E,\Sigma}(\beta)$ implies $k = \ell |w|$ for some $\ell \geq 1$. 

Proof Sketch

Let $\alpha = x_1x_2x_1^2x_2x_1^3x_2^2$ and $\beta = x_3x_4^2x_3^2x_4^6x_3^3$.

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$\Rightarrow$ all solutions to the equations are periodic.

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Since $a^6$ is the shortest element in $L_{E,\Sigma}(\alpha) \cap L_{E,\Sigma}(\beta)$ and $a^8 \in L_{E,\Sigma}(\alpha) \cap L_{E,\Sigma}(\beta)$, we obtain a contradiction.
Other Closure Properties of TF Pattern Languages

Theorem
Let $|\Sigma| \geq 2$. The terminal-free NE- and E-pattern languages, with respect to $\Sigma$, are not closed under
- morphisms,
- inverse morphisms,
- Kleene plus and
- Kleene star.

Theorem
For every terminal-free pattern $\alpha$, the complement of $L_{E,\Sigma}(\alpha)$ is not a terminal-free E-pattern language and the complement of $L_{NE,\Sigma}(\alpha)$ is not a terminal-free NE-pattern language.
Closure Properties of General Pattern Languages

Closure under complement is fully characterised:

**Theorem**

For every pattern $\alpha$, the complement of $L_{E,\Sigma}(\alpha)$ is not an $E$-pattern language and the complement of $L_{NE,\Sigma}(\alpha)$ is not a NE-pattern language.
Main Research Question

For $Z, Z' \in \{E, \text{NE}\}$ and $\circ \in \{\cup, \cap\}$, are there $\alpha, \beta$ such that

- $L_{Z,\Sigma}(\alpha) \circ L_{Z,\Sigma}(\beta)$ is not a $Z'$-pattern language? ✓
- $L_{Z,\Sigma}(\alpha) \circ L_{Z,\Sigma}(\beta)$ is a $Z'$-pattern language?
Main Research Question

For $Z, Z' \in \{E, NE\}$ and $\circ \in \{\cup, \cap\}$, are there $\alpha, \beta$ such that

- $L_{Z, \Sigma}(\alpha) \circ L_{Z, \Sigma}(\beta)$ is not a $Z'$-pattern language? ✓
- $L_{Z, \Sigma}(\alpha) \circ L_{Z, \Sigma}(\beta)$ is a $Z'$-pattern language?

Characterise the $\alpha, \beta$ for which $L_{Z, \Sigma}(\alpha) \circ L_{Z, \Sigma}(\beta)$ is a $Z'$-pattern language?
There are simple examples for the situation that

- \( L_{E,\Sigma}(\alpha) \cap L_{E,\Sigma}(\beta) \) is an E-pattern language.
- \( L_{NE,\Sigma}(\alpha) \cap L_{NE,\Sigma}(\beta) \) is an NE-pattern language.
- \( L_{NE,\Sigma}(\alpha) \cap L_{NE,\Sigma}(\beta) \) is an E-pattern language.
Intersection of General Pattern Languages

There are simple examples for the situation that

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- \( L_{NE,\Sigma}(\alpha) \cap L_{NE,\Sigma}(\beta) \) is an E-pattern language.

Open:

- Are there \( \alpha, \beta \), such that \( L_{E,\Sigma}(\alpha) \cap L_{E,\Sigma}(\beta) \) is NE-pattern language?
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- $L_{E, \Sigma}(\alpha) \cap L_{E, \Sigma}(\beta)$ is an E-pattern language.
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Open:

- Are there $\alpha, \beta$, such that $L_{E, \Sigma}(\alpha) \cap L_{E, \Sigma}(\beta)$ is NE-pattern language?
- Characterisations?
Union of General Pattern Languages

There are simple examples for the situation that

- \( L_{NE,\Sigma}(\alpha) \cup L_{NE,\Sigma}(\beta) \) is an NE-pattern language.
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Union of General Pattern Languages

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Examples for the situation that $L_{E,\Sigma}(\alpha) \cup L_{E,\Sigma}(\beta)$ is an E-pattern language exist, but are much more complicated.
Example for “E ∪ E = E” and alphabet size 2:

Σ = \{a, b\},
\[\alpha = x_1ax_2bx_2ax_3,\]
\[\beta = x_1ax_2bbx_2ax_3,\]
\[\gamma = x_1ax_2bx_3ax_4.\]

\[L_{E, \Sigma}(\alpha) \cup L_{E, \Sigma}(\beta) = L_{E, \Sigma}(\gamma),\]
\[L_{E, \Sigma}(\alpha) \nsubseteq L_{E, \Sigma}(\beta),\]
\[L_{E, \Sigma}(\beta) \nsubseteq L_{E, \Sigma}(\alpha).\]
Example for “E ∪ E = E” and alphabet size 2:

\[ \Sigma = \{a, b\}, \quad \alpha = x_1 ax_2 bx_2 ax_3, \]
\[ \beta = x_1 ax_2 bbx_2 ax_3, \quad \gamma = x_1 ax_2 bx_3 ax_4. \]

Proof sketch:

\[ L_{E,\Sigma}(\alpha) \cup L_{E,\Sigma}(\beta) = L_{E,\Sigma}(\gamma), \]
\[ L_{E,\Sigma}(\alpha) \not\subseteq L_{E,\Sigma}(\beta), \]
\[ L_{E,\Sigma}(\beta) \not\subseteq L_{E,\Sigma}(\alpha). \]
Example for “$E \cup E = E$” and alphabet size 2:

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\alpha = x_1ax_2bx_2ax_3, \\
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\gamma = x_1ax_2bx_3ax_4.
\]

Proof sketch:

$L_{E,\Sigma}(\alpha) \subseteq L_{E,\Sigma}(\gamma)$ and
$L_{E,\Sigma}(\beta) \subseteq L_{E,\Sigma}(\gamma)$ is obvious.
Example for “$E \cup E = E$” and alphabet size 2:

$$\Sigma = \{a, b\},$$
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Let $w \in L_{E,\Sigma}(\gamma)$
Example for “$E \cup E = E$” and alphabet size 2:

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Proof sketch:

$L_{E,\Sigma}(\alpha) \subseteq L_{E,\Sigma}(\gamma)$ and
$L_{E,\Sigma}(\beta) \subseteq L_{E,\Sigma}(\gamma)$ is obvious.

Let $w \in L_{E,\Sigma}(\gamma)$
$w = u \ a \ b^n \ a \ v$, 

$\Sigma = \{ a, b \}$,
$\alpha = x_1 ax_2 bx_2 ax_3$,
$\beta = x_1 ax_2 bbx_2 ax_3$,
$\gamma = x_1 ax_2 bx_3 ax_4$. 

$L_{E,\Sigma}(\alpha) \cup L_{E,\Sigma}(\beta) = L_{E,\Sigma}(\gamma),$
$L_{E,\Sigma}(\alpha) \not\subseteq L_{E,\Sigma}(\beta),$
$L_{E,\Sigma}(\beta) \not\subseteq L_{E,\Sigma}(\alpha)$. 

Union of General Pattern Languages
Example for “E ∪ E = E” and alphabet size 2:

\[ \Sigma = \{ a, b \}, \]
\[ \alpha = x_1 ax_2 bx_2 ax_3, \]
\[ \beta = x_1 ax_2 bx_2 bx_2 ax_3, \]
\[ \gamma = x_1 ax_2 bx_3 ax_4. \]

Proof sketch:

\[ L_{E, \Sigma}(\alpha) \subseteq L_{E, \Sigma}(\gamma) \] and
\[ L_{E, \Sigma}(\beta) \subseteq L_{E, \Sigma}(\gamma) \] is obvious.

Let \( w \in L_{E, \Sigma}(\gamma) \)
\[ w = u a b^n a v, \]
\( n \) is even \( \Rightarrow w \in L_{E, \Sigma}(\beta). \)
Example for “E ∪ E = E” and alphabet size 2:

\[ \Sigma = \{a, b\}, \]
\[ \alpha = x_1ax_2bx_2ax_3, \]
\[ \beta = x_1ax_2bbx_2ax_3, \]
\[ \gamma = x_1ax_2bx_3ax_4. \]

\[ L_E,\Sigma(\alpha) \cup L_E,\Sigma(\beta) = L_E,\Sigma(\gamma), \]
\[ L_E,\Sigma(\alpha) \nsubseteq L_E,\Sigma(\beta), \]
\[ L_E,\Sigma(\beta) \nsubseteq L_E,\Sigma(\alpha). \]

Proof sketch:

\[ L_E,\Sigma(\alpha) \subseteq L_E,\Sigma(\gamma) \text{ and } \]
\[ L_E,\Sigma(\beta) \subseteq L_E,\Sigma(\gamma) \text{ is obvious.} \]

Let \( w \in L_E,\Sigma(\gamma) \)
\[ w = uab^nava, \]
\[ n \text{ is even } \Rightarrow w \in L_E,\Sigma(\beta). \]
\[ n \text{ is odd } \Rightarrow w \in L_E,\Sigma(\alpha). \]
Example for “$E \cup E = E$” and alphabet size 3:

$\Sigma = \{a, b, c\}$,

$\alpha = x_1 a x_2 x_3^6 x_4 x_5^6 x_6 b x_7 a x_2 x_8 x_9^{12} x_4 x_5^{12} x_6 b x_{10}$,

$\beta = x_1 a x_2 x_3^6 x_4^2 x_5^5 x_6 x_7 b x_8 a x_2 x_9^{12} x_4 x_5^{10} x_{10}^{12} x_7 b x_{11}$,

$\gamma = x_1 a x_2 x_3^6 x_4^2 x_5^3 x_6 x_7 b x_8 a x_2 x_9^{12} x_4 x_5^{6} x_{10}^{12} x_7 b x_{11}$.

$L_{E, \Sigma}(\alpha) \cup L_{E, \Sigma}(\beta) = L_{E, \Sigma}(\gamma),$

$L_{E, \Sigma}(\alpha) \not\subseteq L_{E, \Sigma}(\beta),$

$L_{E, \Sigma}(\beta) \not\subseteq L_{E, \Sigma}(\alpha)$. 
Union of General Pattern Languages

Example for “$E \cup E = E$” and alphabet size 4:

$\Sigma = \{a, b, c, d\}$,

$\alpha := x_1 ax_2 x_3^2 x_4^2 x_5 x_6 b x_7 a x_2 x_8^2 x_4 x_9 x_6 b$
$\qquad x_{10} c x_{11} x_{12}^2 x_{13}^2 x_{14}^2 x_{15} x_{16} d x_{17} c x_{11} x_{18}^2 x_{13}^2 x_{14}^2 x_{19} x_{16} d$
$\qquad x_{20} x_{13}^2 x_{14}^2 x_{13}^2 x_{14}^2 x_{19} x_{14} x_{21} x_{6}$,

$\beta := x_1 ax_2 x_3^2 x_4^2 x_5^2 x_6^2 x_7 b x_8 a x_2 x_9^2 x_4 x_5^2 x_10^2 x_7 b$
$\qquad x_{11} c x_{12} x_{13}^2 x_{14}^2 x_{16} d x_{17} c x_{12} x_{18}^2 x_{14}^2 x_{19} x_{16} d$
$\qquad x_{20} x_{14}^6 x_{21} x_{4}^2 x_{5}^2 x_{4}^2 x_{5}^2 x_{4}^2 x_{5}^2$ and

$\gamma := x_1 ax_2 x_3^2 x_4^2 x_5^2 x_6^2 x_7 b x_8 a x_2 x_9^2 x_4 x_5^2 x_10^2 x_7 b$
$\qquad x_{11} c x_{12} x_{13}^2 x_{14}^2 x_{15} x_{16} d x_{17} c x_{12} x_{19}^2 x_{14}^2 x_{15}^2 x_{20} x_{17} d$
$\qquad x_{21} x_{14}^2 x_{15}^2 x_{14}^2 x_{15}^2 x_{15}^2 x_{22}^2 x_{4}^2 x_5^2 x_4^2 x_5^2 x_4^2 x_5^2$.

$L_{E,\Sigma}(\alpha) \cup L_{E,\Sigma}(\beta) = L_{E,\Sigma}(\gamma)$, $L_{E,\Sigma}(\alpha) \not\subseteq L_{E,\Sigma}(\beta)$, $L_{E,\Sigma}(\beta) \not\subseteq L_{E,\Sigma}(\alpha)$. 
Necessary Condition for $E \cup E = E$

\[ \alpha = \alpha_0 u_1 \alpha_1 u_2 \alpha_2 \cdots \alpha_{n-1} u_n, \]
\[ \beta = \beta_0 v_1 \beta_1 v_2 \beta_2 \cdots \beta_{m-1} v_m, \]
\[ \gamma = \gamma_0 w_1 \gamma_1 w_2 \gamma_2 \cdots \gamma_{m-1} w_k, \]
\[ \alpha_i, \beta_i, \gamma_i \in X^+, \ u_i, v_i, w_i \in \Sigma^+. \]
Necessary Condition for $E \cup E = E$

$\alpha = \alpha_0 u_1 \alpha_1 u_2 \alpha_2 \ldots \alpha_{n-1} u_n,$
$\beta = \beta_0 v_1 \beta_1 v_2 \beta_2 \ldots \beta_{m-1} v_m,$
$\gamma = \gamma_0 w_1 \gamma_1 w_2 \gamma_2 \ldots \gamma_{m-1} w_k,$
$\alpha_i, \beta_i, \gamma_i \in X^+, u_i, v_i, w_i \in \Sigma^+.$

$L_{E,\Sigma}(\alpha) \cup L_{E,\Sigma}(\beta) = L_{E,\Sigma}(\gamma)$

$\implies$

$w_0 w_1 \ldots w_k = u_0 u_1 \ldots u_k$ and $w_0 w_1 \ldots w_k$ subsequence of $v_0 v_1 \ldots v_k$ or
$w_0 w_1 \ldots w_k = v_0 v_1 \ldots v_k$ and $w_0 w_1 \ldots w_k$ subsequence of $u_0 u_1 \ldots u_k$
Necessary Condition for $\text{NE} \cup \text{NE} = \text{NE}$

Let $\{a, b\} \subseteq \Sigma$, let $\alpha$, $\beta$ and $\gamma$ be patterns with neither

- $L_{\text{NE},\Sigma}(\alpha) \subseteq L_{\text{NE},\Sigma}(\beta)$, $\beta = \gamma$ nor
- $L_{\text{NE},\Sigma}(\beta) \subseteq L_{\text{NE},\Sigma}(\alpha)$, $\alpha = \gamma$. 
Necessary Condition for $NE \cup NE = NE$

Let $\{a, b\} \subseteq \Sigma$, let $\alpha$, $\beta$ and $\gamma$ be patterns with neither

- $L_{NE,\Sigma}(\alpha) \subseteq L_{NE,\Sigma}(\beta)$, $\beta = \gamma$ nor
- $L_{NE,\Sigma}(\beta) \subseteq L_{NE,\Sigma}(\alpha)$, $\alpha = \gamma$.

Let $\alpha = \delta_0 a \delta_1 a \delta_2 \ldots \delta_{m-1} a \delta_m$, $\beta = \delta_0 b \delta_1 b \delta_2 \ldots \delta_{m-1} b \delta_m$, $\gamma = \delta_0 x \delta_1 x \delta_2 \ldots \delta_{m-1} x \delta_m$, where $m \geq 1$, $\delta_i \in (X \cup \Sigma)^*$, $0 \leq i \leq m$. 

\[
L_{NE,\Sigma}(\alpha) \cup L_{NE,\Sigma}(\beta) = L_{NE,\Sigma}(\gamma)
\]
Characterisations for $\text{NE} \cup \text{NE} = \text{E}$

Let $|\Sigma| \geq 2$, let $\alpha$, $\beta$ and $\gamma$ be patterns.
Characterisations for $\text{NE} \cup \text{NE} = E$

Let $|\Sigma| \geq 2$, let $\alpha$, $\beta$ and $\gamma$ be patterns.

$$L_{\text{NE}, \Sigma}(\alpha) \cup L_{\text{NE}, \Sigma}(\beta) = L_{E, \Sigma}(\gamma)$$

$\iff$

$\alpha = u_1 u_2 \ldots u_{m+1} \in \Sigma^+$ and $\beta = \gamma = u_1 x^{j_1} u_2 x^{j_2} \ldots x^{j_m} u_{m+1}, j_i \in \mathbb{N}_0$. 
Characterisations for \( \text{NE} \cup \text{NE} = \text{E} \)

Let \(|\Sigma| \geq 2\), let \(\alpha, \beta\) and \(\gamma\) be patterns.

Let \(\alpha = u_1 \ u_2 \ldots u_{m+1} \in \Sigma^+ \) and \(\beta = \gamma = u_1 \ x_1^{j_1} \ u_2 \ x_2^{j_2} \ldots \ x_m^{j_m} \ u_{m+1}, j_i \in \mathbb{N}_0\).

This corresponds to the canonical way of expressing \(\text{E}\)-pattern languages by unions of \(\text{NE}\)-pattern languages.
Characterisations for $E \cup E = NE$

Let $\{a_1, a_2, \ldots, a_\ell\} \subseteq \Sigma$, $\ell \geq 2$, let $\alpha_1, \alpha_2, \ldots, \alpha_\ell, \gamma$ be patterns with $L_{E,\Sigma}(\alpha_i) \neq L_{E,\Sigma}(\alpha_j)$. 
Characterisations for $E \cup E = NE$

Let $\{a_1, a_2, \ldots, a_\ell\} \subseteq \Sigma$, $\ell \geq 2$, let $\alpha_1, \alpha_2, \ldots, \alpha_\ell$, $\gamma$ be patterns with $L_{E, \Sigma}(\alpha_i) \neq L_{E, \Sigma}(\alpha_j)$.

\[
\bigcup_{i=1}^{\ell} L_{E, \Sigma}(\alpha_i) = L_{NE, \Sigma}(\gamma)
\]

$\iff$

$\Sigma = \{a_1, a_2, \ldots, a_\ell\}$

$\gamma = u_1 \times u_2 \times u_3 \ldots u_k \times u_{k+1}$

$\alpha_i = u_1 \alpha'_i a_i \alpha''_i u_2 \alpha'_i a_i \alpha''_i u_3 \ldots u_k \alpha'_i a_i \alpha''_i u_{k+1}$

$\alpha'_i, \alpha''_i \in X^*$ and, there exists a variable $y_i$ with exactly one occurrence in $\alpha'_i a_i \alpha''_i$. 

Thank you very much for your attention.