# Graph and String Parameters: Connections Between Pathwidth, Cutwidth and the Locality Number

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# A Solitaire Game on Strings

### The Game

Given: String $\alpha$ over (finite) alphabet $\Sigma = \{a_1, a_2, \ldots, a_n\}$.

Objective: Mark all symbols $a_1, a_2, \ldots, a_n$ in some order (all occ. of the same symbol in parallel), such that there are only few contiguous blocks of marked symbols in the word.

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## Example

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marking sequence:

marked blocks:
maximum number of marked blocks:

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## Example

`a d a b a d b d a e c b c b`

marking sequence: `b`

marked blocks: 4
maximum number of marked blocks: 4

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## Example

a d a b a d b d a e c b c b

marking sequence: b, c

marked blocks: 3
maximum number of marked blocks: 4

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## Example

a d a b a d b d a e c b c b

marking sequence: b, c, e, d, a

marked blocks: 1
maximum number of marked blocks: 4

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marking sequence: `d`

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## Example

`a d a b a d b d a e c b c b`

marking sequence: `d`, `a`, `b`, `c`

marked blocks: 2
maximum number of marked blocks: 3

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## Example

`a d a b a d b d a e c b c b`

marking sequence: `d`, `a`, `b`, `c`, `e`

marked blocks: 1
maximum number of marked blocks: 3

# The Locality Number

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite alphabet.

## Marking Sequence

Any ordered list $\sigma$ of the symbols from $X$ (or, equivalently, a bijection $\sigma : \{1, 2, \ldots, |X|\} \to X$) is a **marking sequence**.

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## Locality Number

A string $\alpha$ over $X$ is **$k$-local** $\iff \pi_\sigma(\alpha) \leq k$, for some marking sequence $\sigma$.

The **locality number** of $\alpha$ is $\mathrm{loc}(\alpha) = \min\{k \mid \alpha \text{ is } k\text{-local}\}$.

# The Locality Number

## Example

Let $\alpha = \texttt{adabadbdaecbcb}$, $\sigma_1 = (\texttt{b}, \texttt{c}, \texttt{e}, \texttt{d}, \texttt{a})$, $\sigma_2 = (\texttt{d}, \texttt{a}, \texttt{b}, \texttt{c}, \texttt{e})$

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## Motivation

Pattern matching with variables.

Marking sequence = dynamic programming algorithm

$\rightsquigarrow$ XP-algorithms w.r.t. parameter $\mathsf{loc}(\alpha)$.

# Known Results and Open Problems

## Computing the locality number

Loc

| | |
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| Input: | String $\alpha \in \Sigma^*$, $k \in \mathbb{N}$. |
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## Open Problems

- Is Loc NP-complete?
- Is Loc $\in$ FPT (w.r.t. $k$ or $|\Sigma|$)?
- Are there good approximation algorithms for MinLoc?

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Let $G = (V, E)$ be a (multi)graph with $V = \{v_1, \ldots, v_n\}$.

### Cuts

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## Linear Arrangements and Cutwidth

**Linear arrangement of $G$**: sequence $L = (v_{j_1}, v_{j_2}, \ldots, v_{j_n})$, where $(j_1, j_2, \ldots, j_n)$ is a permutation of $(1, 2, \ldots, n)$.

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$\mathrm{cw}(L) = \max\{|\mathcal{C}(\{v_{j_1}, v_{j_2}, \ldots, v_{j_i}\}, \{v_{j_{i+1}}, \ldots, v_{j_n}\})| \mid 0 \le i \le n\}$

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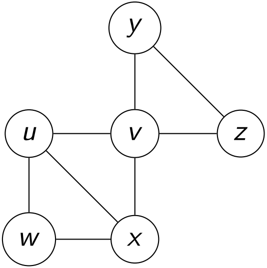
**Cutwidth of $G$**: $\text{cw}(G) = \min\{\text{cw}(L) \mid L \text{ is lin. arr. for } G\}$.

# Cutwidth – Example

Graph $G$:



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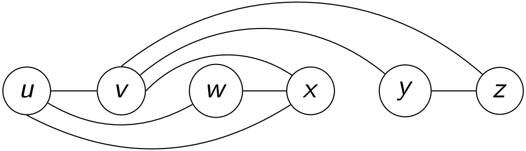
Linear arrangement with cutwidth 5:

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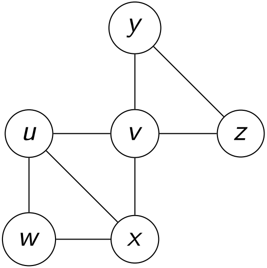

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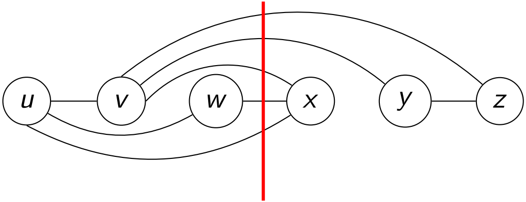


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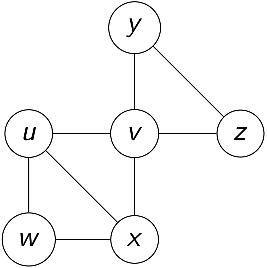

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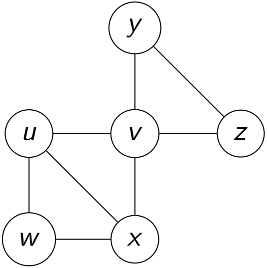


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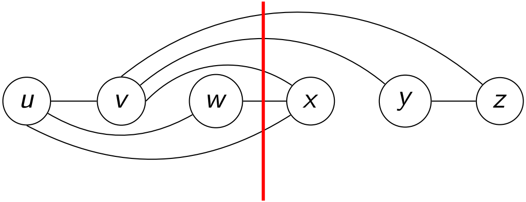


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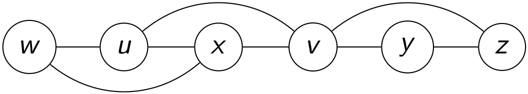

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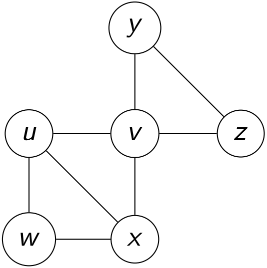


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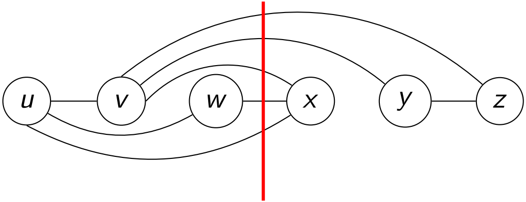


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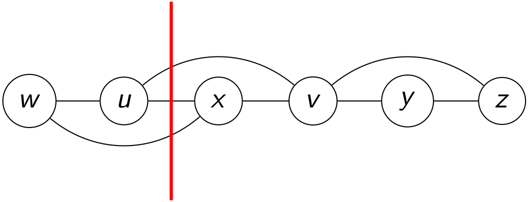

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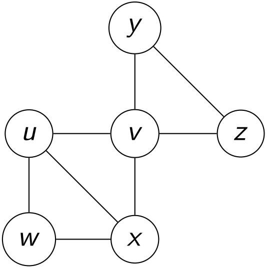


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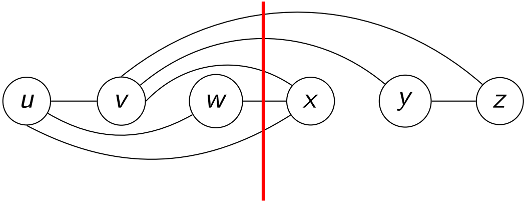


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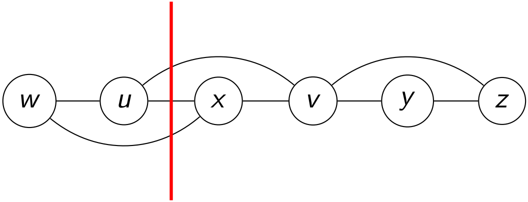


$\text{cw}(G) = 3$

# Computing the Cutwidth

## Cutwidth problem

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| Input: | (Multi)graph $G$, $k \in \mathbb{N}$. |
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MinCutwidth denotes the corresponding minimisation problem.

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## Known Results

- Cutwidth is NP-complete.
- Cutwidth $\in$ FPT (w.r.t. $k$).
- Exact exponential algorithms, linear fpt-algorithms, approximation algorithms...

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$\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$
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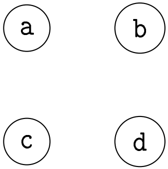

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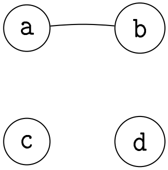

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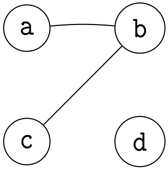

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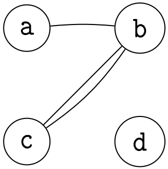

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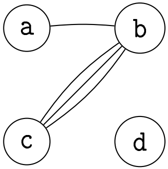

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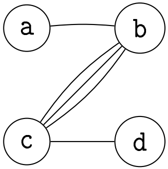

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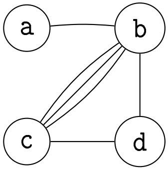

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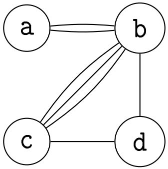

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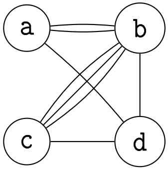
Construct multigraph $H_{\alpha,k} = (V, E)$:



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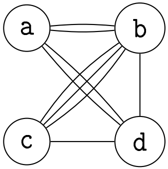

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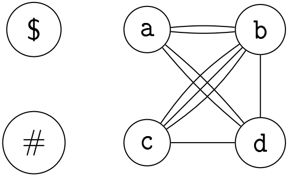

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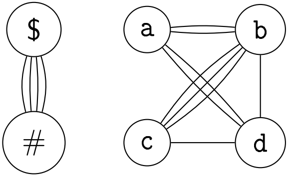

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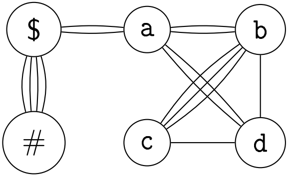

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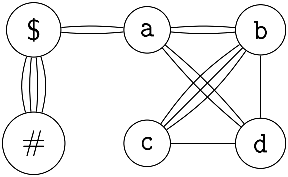


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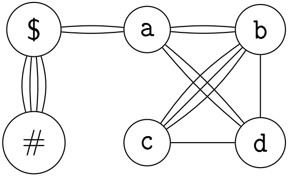


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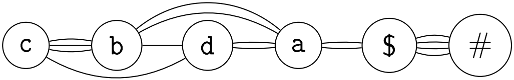

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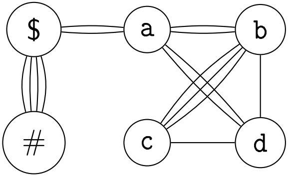


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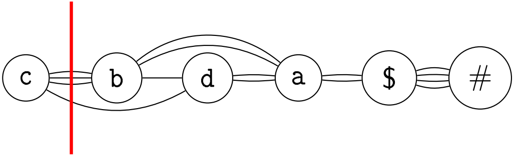

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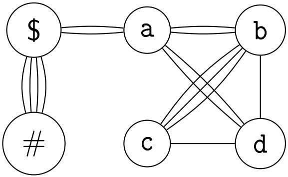


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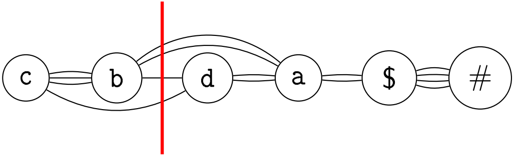

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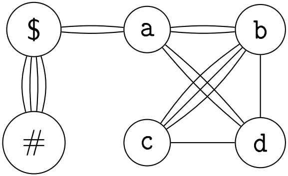


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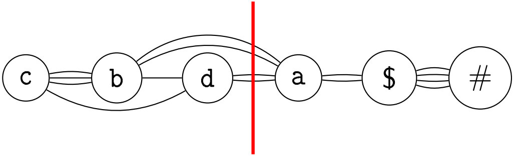

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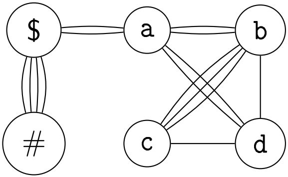


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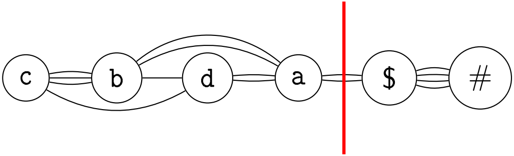

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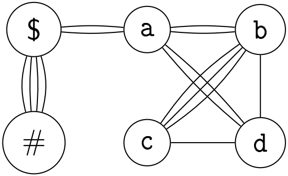


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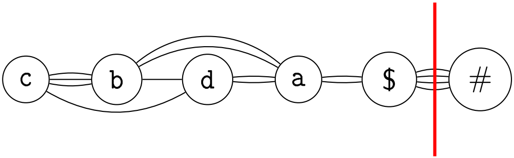

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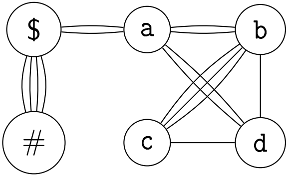


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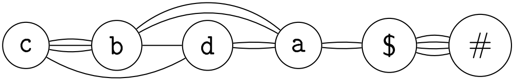

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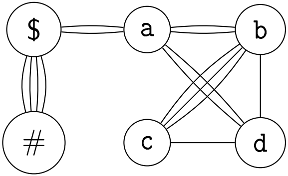


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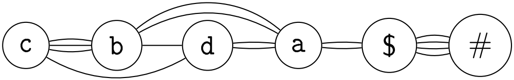


**Lemma**

$\mathrm{cw}(H_{\alpha,k}) = 2k$ if and only if $\mathrm{loc}(\alpha) \leq k$.

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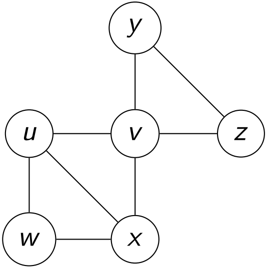

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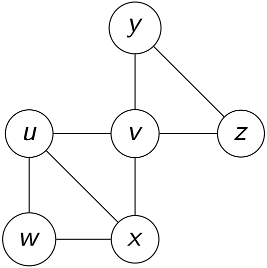

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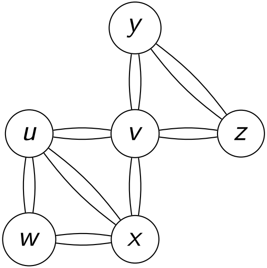

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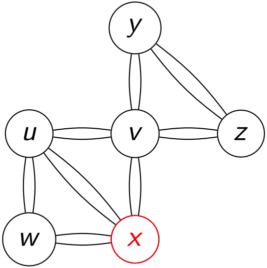


$x$

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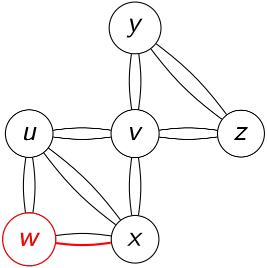


$x$ $w$

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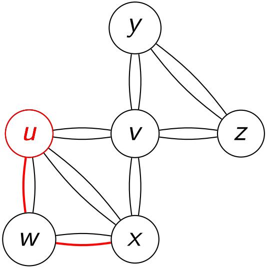


*x w* *u*

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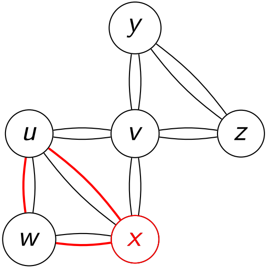


$x\ w\ u\ x$

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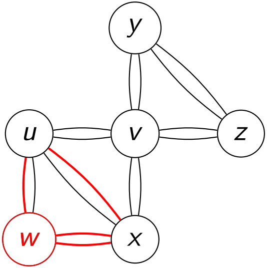


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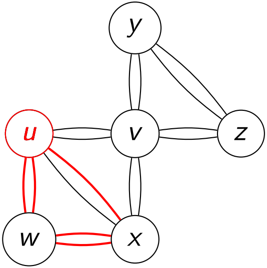


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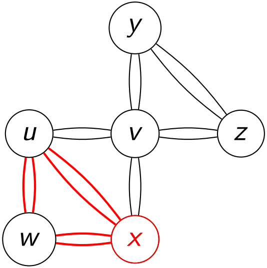


$x\ w\ u\ x\ w\ u\ x$

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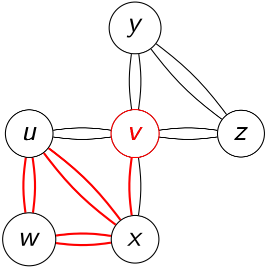


$x\ w\ u\ x\ w\ u\ x\ v$

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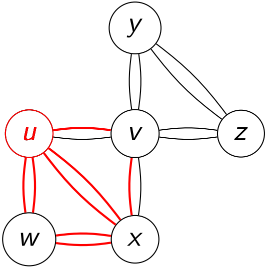


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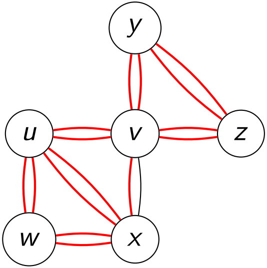


$x \; w \; u \; x \; w \; u \; x \; v \; u \; v \; y \; z \; v \; y \; z \; v$

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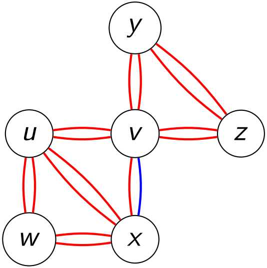


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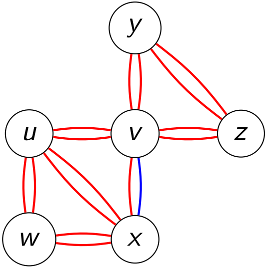


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**Lemma**

$\forall e \in E : \mathrm{cw}(G) \leq \mathrm{loc}(\alpha_e) \leq \mathrm{cw}(G) + 1$

$\exists e \in E : \mathrm{loc}(\alpha_e) = \mathrm{cw}(G).$

# Consequences

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## Approximation Meta-Theorem

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# Path-Decompositions and Pathwidth

**Path-decompositions as tree-decomposition**

A path-decomposition is a tree-decomposition the underlying tree-structure of which is a path.

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**pw**$(G)$: Min. pw$(Q)$ over all path-decompositions.

# Computing the Pathwidth

## Pathwidth problem

Pathwidth

| | |
|---|---|
| Input: | Graph $G$, $k \in \mathbb{N}$. |
| Question: | $\mathrm{pw}(\alpha) \leq k$? |

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MinPathwidth denotes the corresponding minimisation problem.

## Known Results

- Pathwidth is NP-complete.
- Pathwidth $\in$ FPT (w.r.t. $k$).
- Exact exponential algorithms, linear fpt-algorithms, approximation algorithms...

# Loc $\leq$ Pathwidth

$\alpha = \texttt{c a b a c a b a c}$

# Loc $\leq$ Pathwidth

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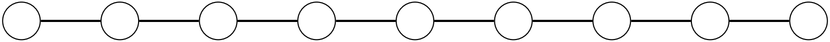
$\mathcal{G}_\alpha$:



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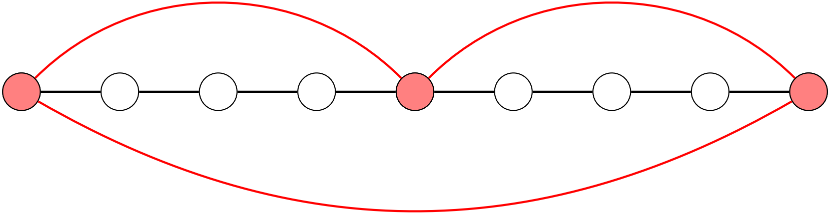

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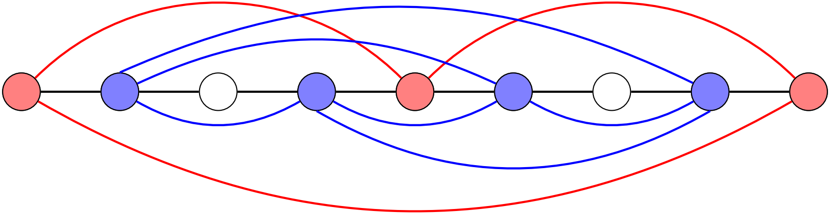

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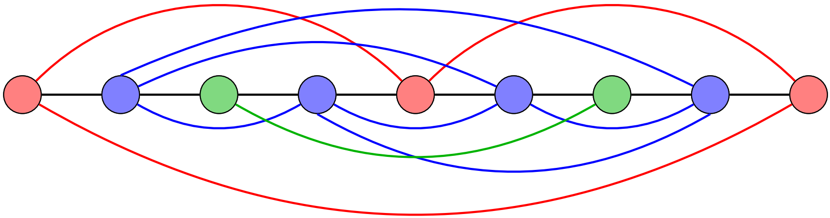

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### Theorem

There is an $O(\sqrt{\log(\mathrm{opt})}\log(n))$-approx. algo. for MinLoc.

# Consequences for Cutwidth

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Leighton and Rao, JACM 1999: $O(\log(n)\log(n))$-approximation.

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### MinCutwidth $\leq$ MinLoc $\leq$ MinPathwidth

There is an $O(\sqrt{\log(\text{opt})}\log(h))$-approximation algorithm for MinCutwidth on multigraphs with $h$ edges.

# Direct Reduction: MinCutwidth $\leq$ MinPathwidth