Characterising REGEX Languages by Regular Languages Equipped with Factor-Referencing

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regulated rewriting (add control mechanisms to context-free grammars)

mildly context-sensitive (allow only a little bit of context-sensitivity).

Typical Non-Context-Free Features

Reduplication	$\{ww \mid w \in \Sigma^*\}$
Multiple agreements	$\{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 1\}$
Crossed agreements	$\{\mathtt{a}^n\mathtt{b}^m\mathtt{c}^n\mathtt{d}^m \mid n, m \geq 1\}$

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Language descriptor/grammars models tailored to reduplication:

- L systems,
- pattern languages,
- H-systems,
- Wijngaarden grammars, macro grammars, Indian parallel grammars, deterministic iteration grammars,
- pattern expressions, synchronized regular expressions, EH-expressions, extended regular expressions with backreferences (REGEX).

Reference-Words - Idea

$$a\;\underbrace{b\;a\;\rlap{$\overbrace{c\;b}^{y}$}c}_{x}\;\;c\;\overbrace{x\;c\;b}^{z}\;z\;y\;a$$

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$$\mathtt{a}\;\underbrace{\mathtt{b\;a\;c}}_{x}\;\mathtt{b}\;\mathtt{c}\;\overbrace{x\;\mathtt{c}\;\mathtt{b}}^{z}\;z\;y\;\mathtt{a}$$

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`a b a c b c b a c c b b a c c b c b a`

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  - only well-formed, non-nested pairs of parentheses $[_{x_i},]_{x_i}$,
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- $\Sigma^{[*]}$ is the set of valid ref-words (over Σ).

Reference-Words - Examples

- $[_x\ [_y\ \mathtt{b}\ ]_x\ \mathtt{c}x\ [_x\ \mathtt{b}\ ]_y\ zy\mathtt{b}\ [_y\ \mathtt{c}z\ ]_y\ z\ [_z\ \mathtt{cc}\ ]_x\ ]_z$,
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References for x with a value u: $[_x u]_x$.

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Overlapping references: $[_x \ldots [_y \ldots]_x \ldots]_y$.

Reference-Words - Dereference Function

$\mathcal{D} : \Sigma^{[*]} \to \Sigma^*$

Example:

$$z\mathtt{a}[_z x[_x y\mathtt{b}[_y \mathtt{c}]_x \mathtt{b} x[_x \mathtt{c}]_y \mathtt{b}]_x y\mathtt{c}]_z x\mathtt{c}z$$

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Ref-Languages and Ref-Regular Languages

L is a ref-language (over Σ) if

- $L \subseteq \Sigma^{[*]}$ and
- $L \subseteq (\Sigma \cup \{[_{x_i},]_{x_i}, x_i \mid i \leq k\})^*$, for some $k \in \mathbb{N}$.

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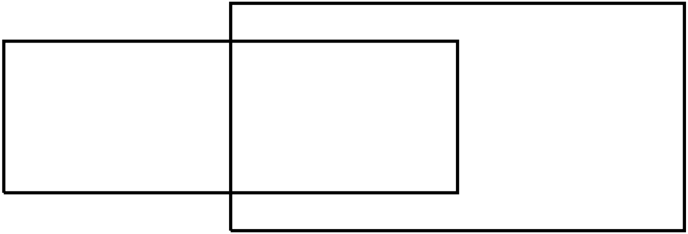
ref-lang.



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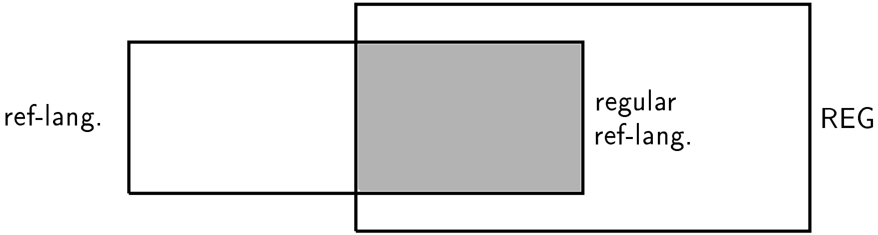

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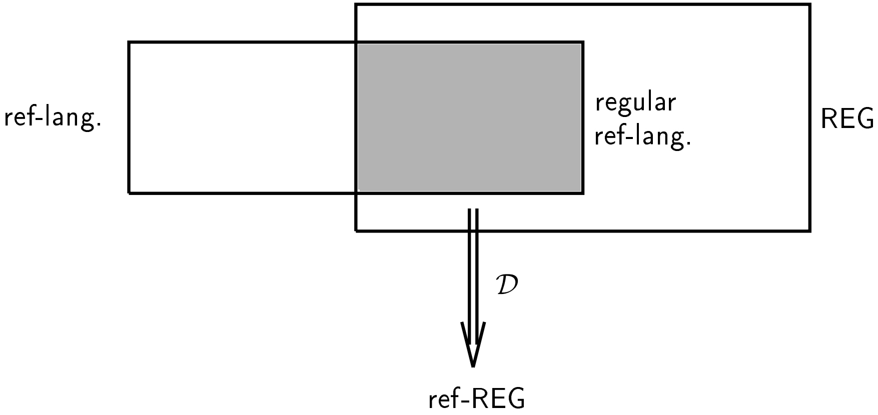

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- $\{\mathtt{a}^n\mathtt{b}^n \mid n \in \mathbb{N}\} \notin$ ref-REG.

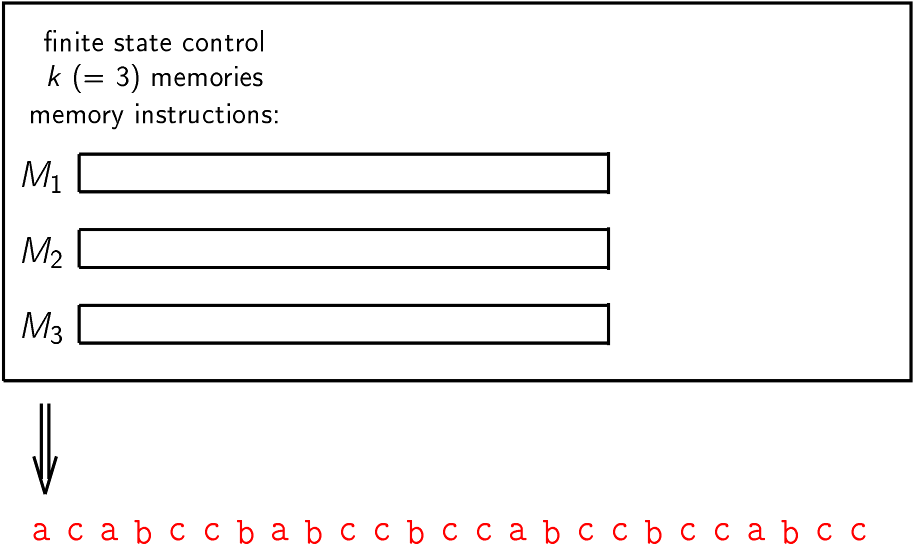
Memory Automata (MFA)

finite state control

k $(= 3)$ memories

memory instructions:

Memory Automata (MFA)



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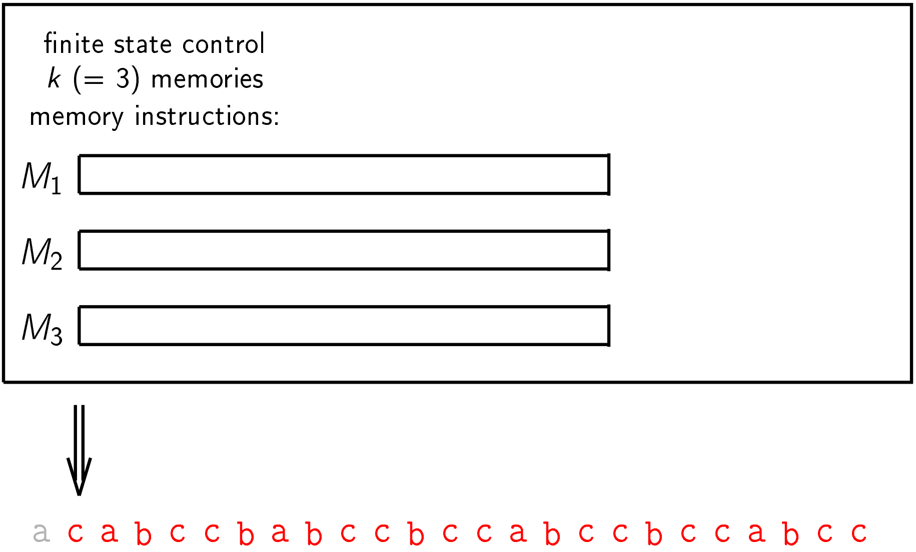

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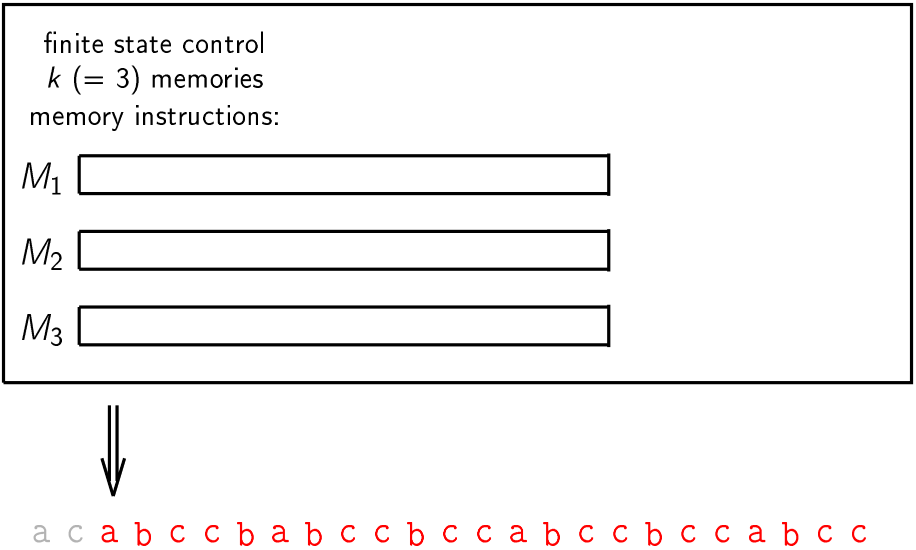

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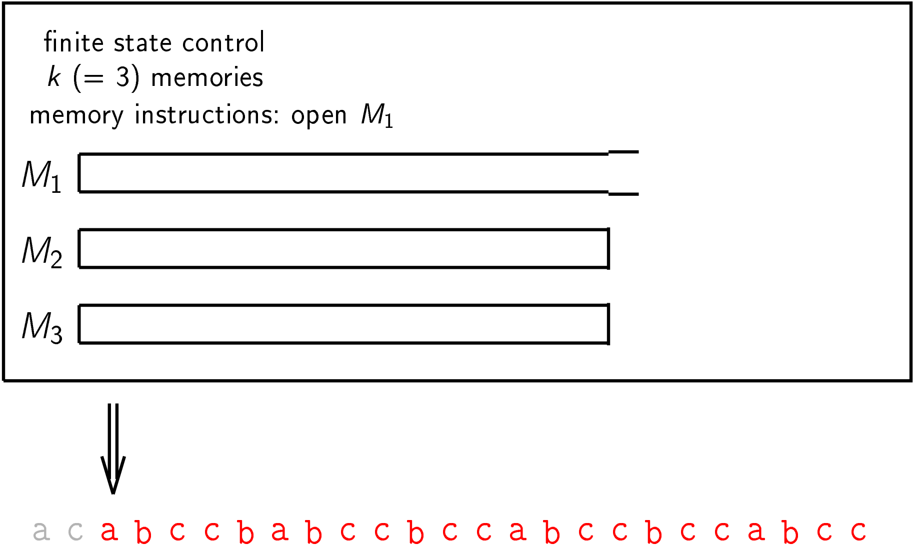

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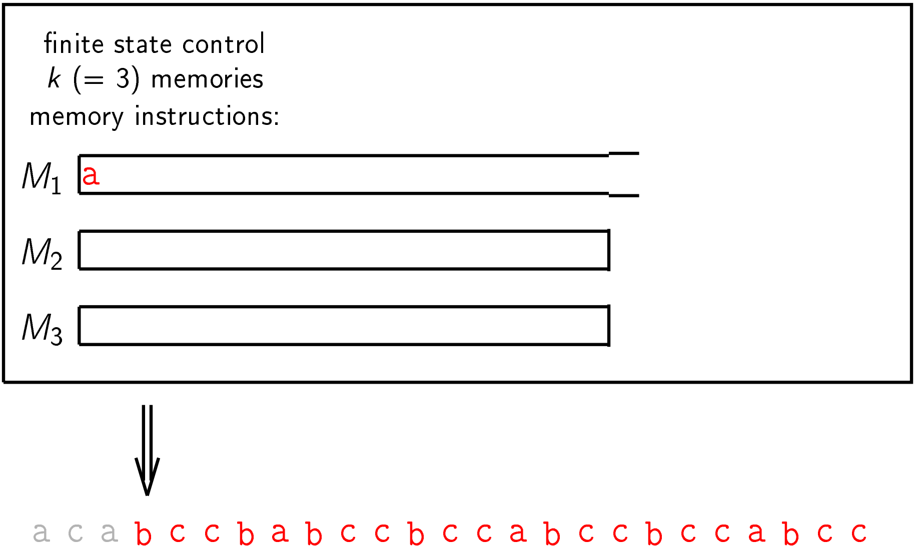

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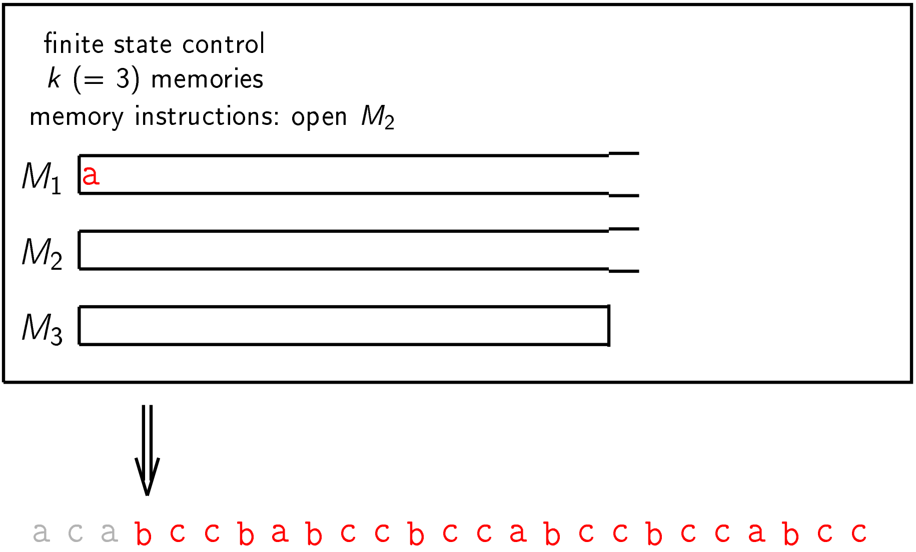

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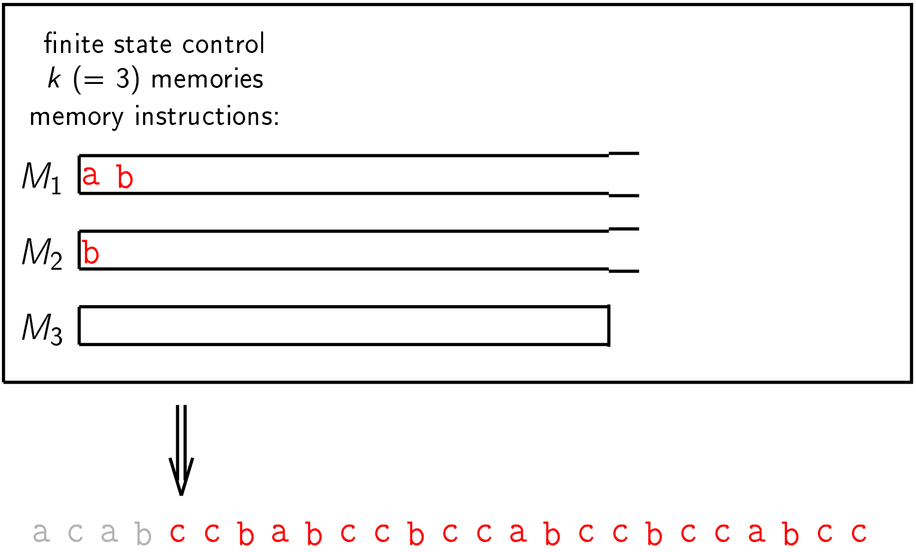

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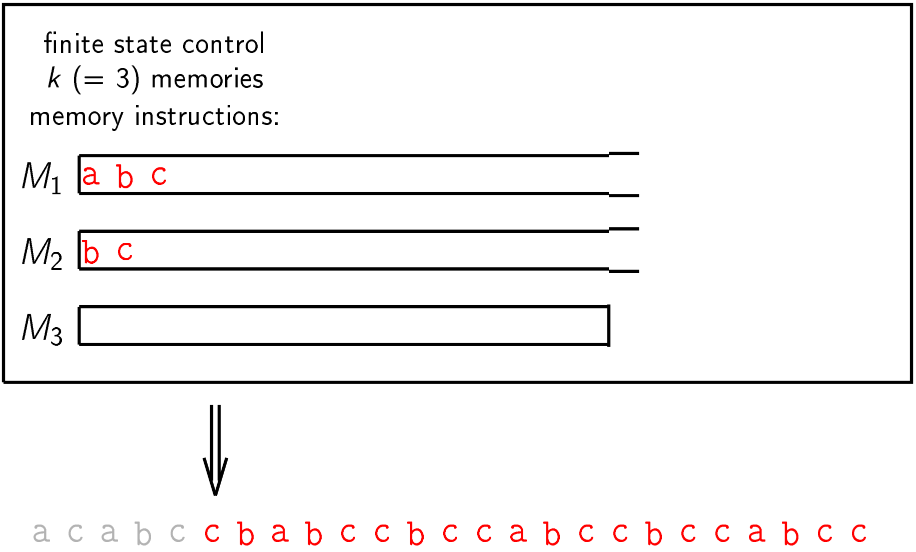

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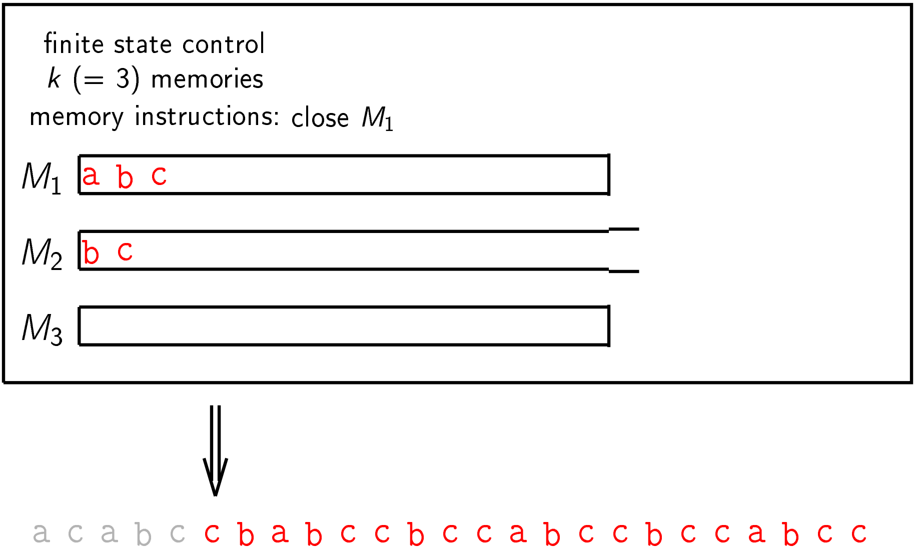

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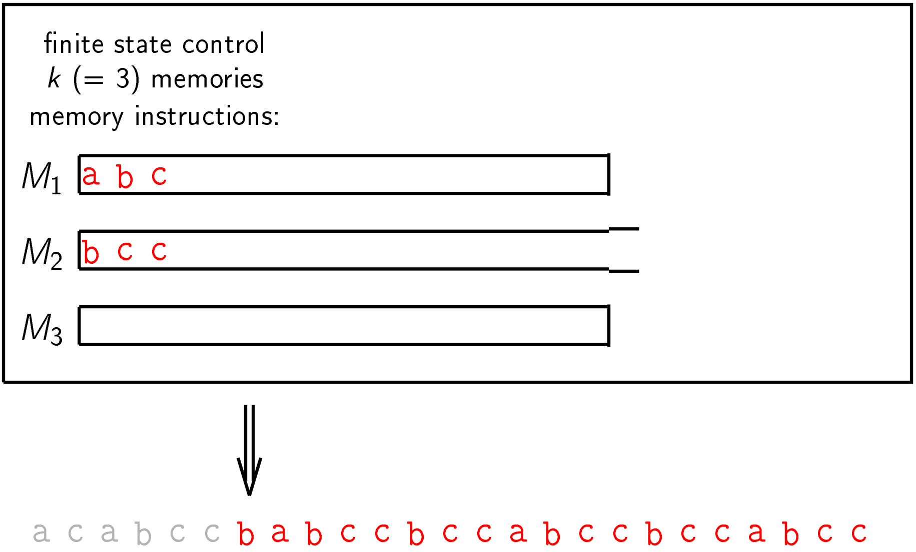

Memory Automata (MFA)

finite state control

k (= 3) memories

memory instructions: close M_2

M_1 a b c

M_2 b c c

M_3

$\Downarrow$

a c a b c c b a b c c b c c a b c c b c c a b c c

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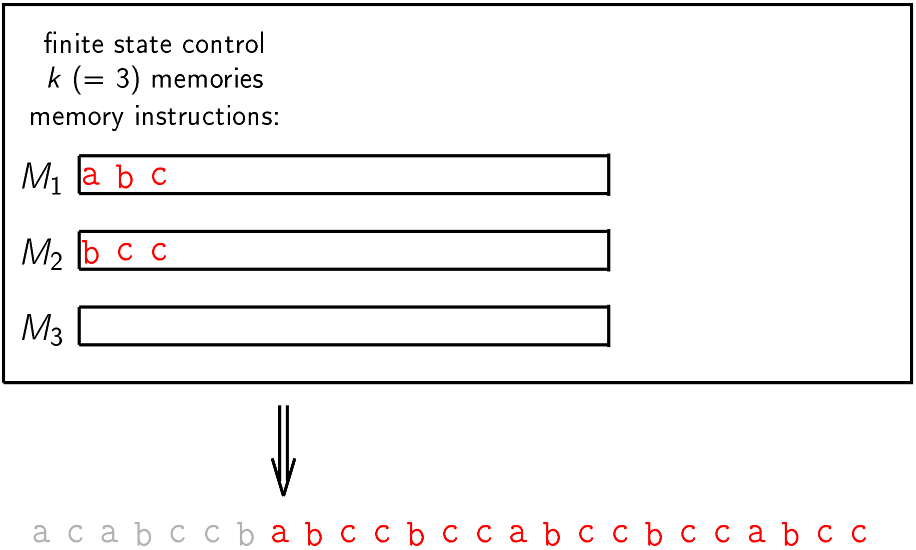

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M_1 a b c

M_2 b c c

M_3

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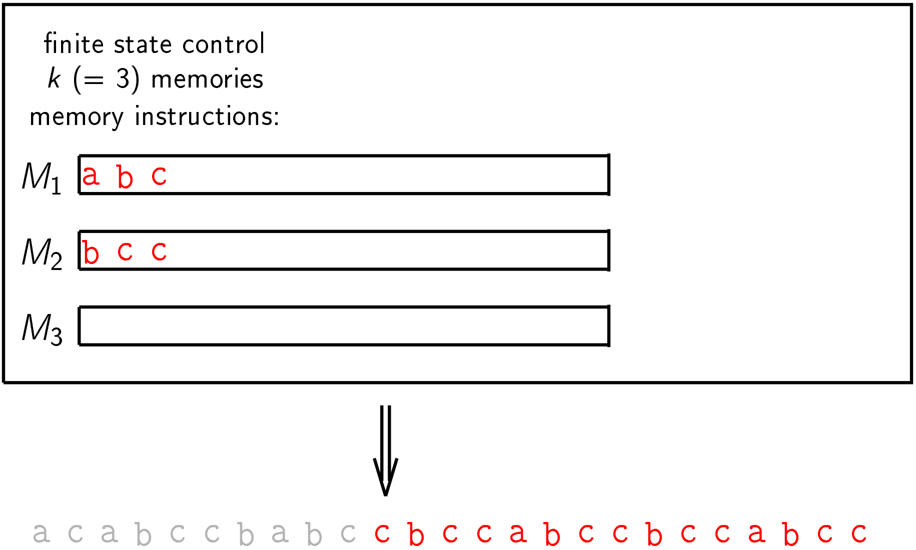

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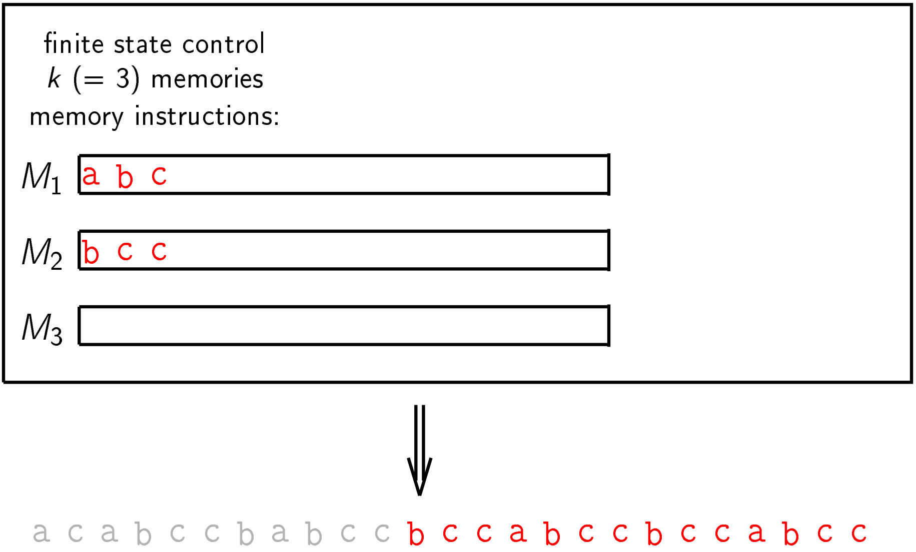

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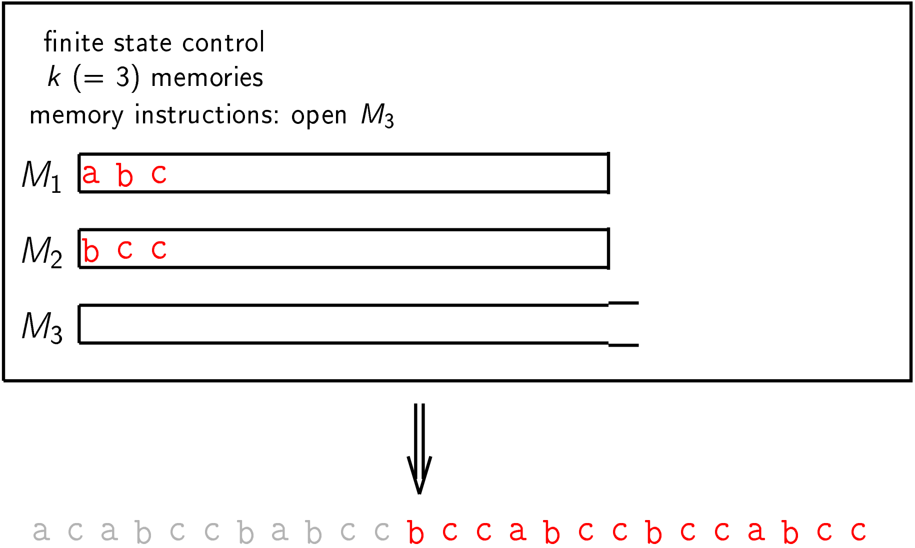

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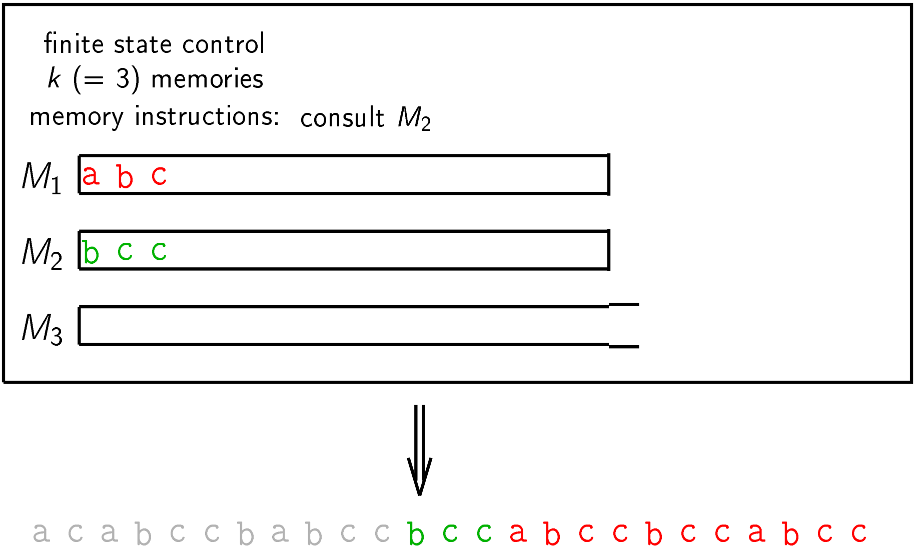

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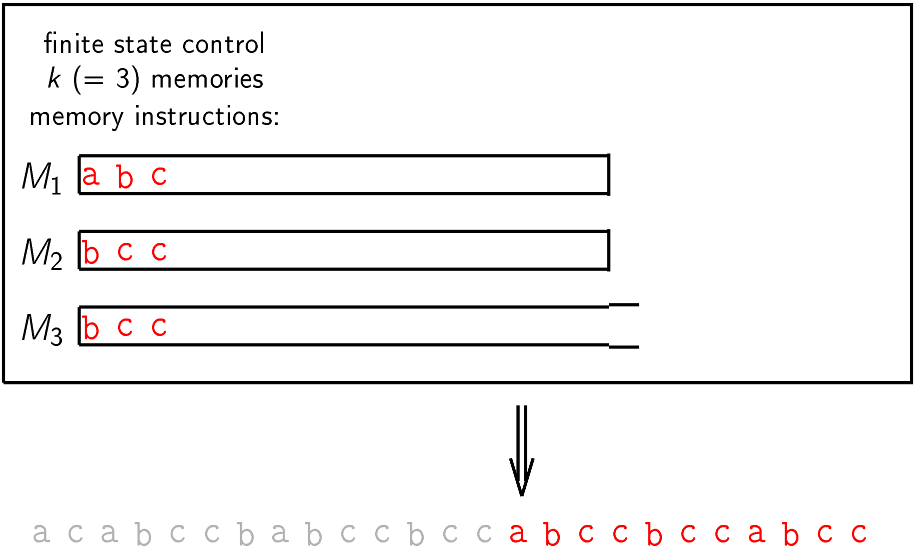

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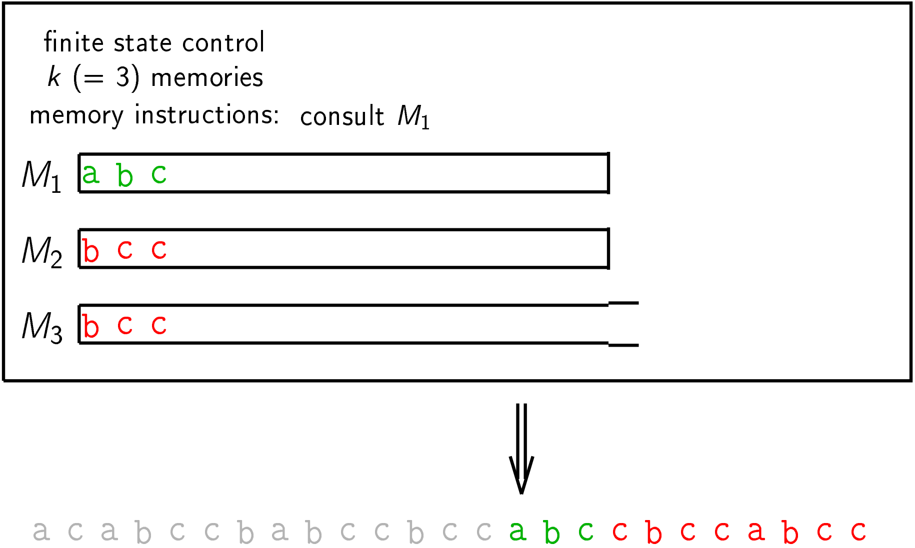

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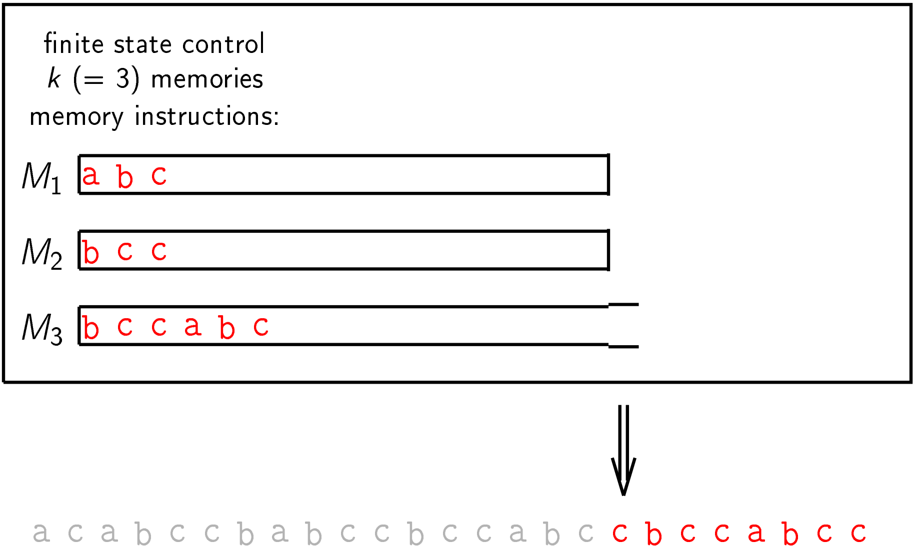

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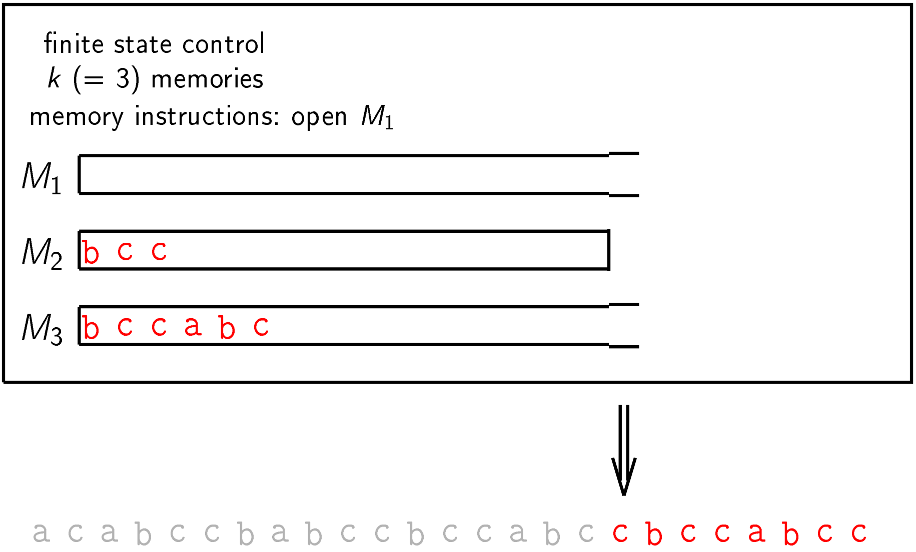

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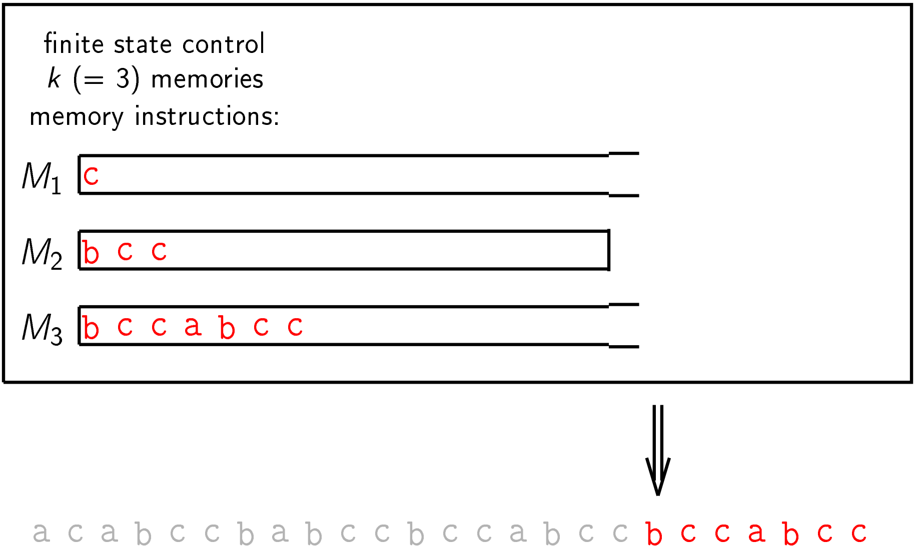

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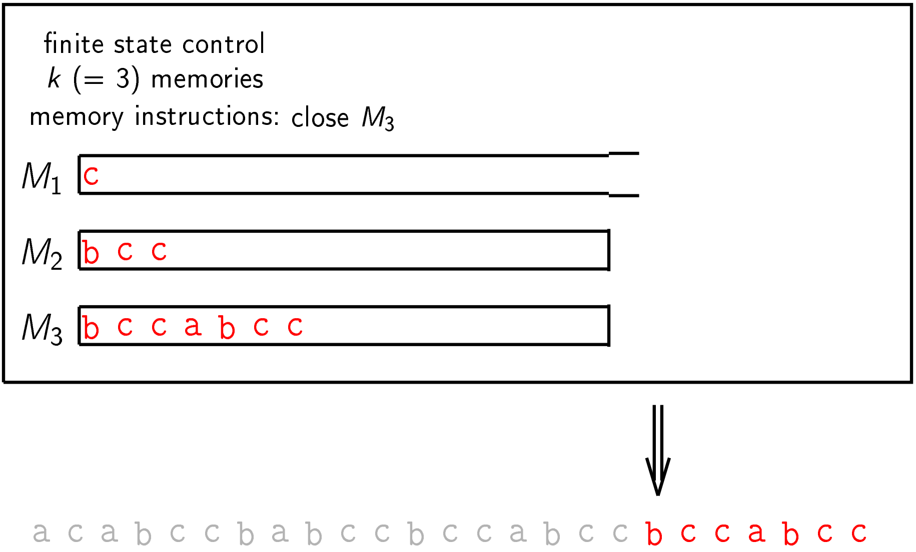

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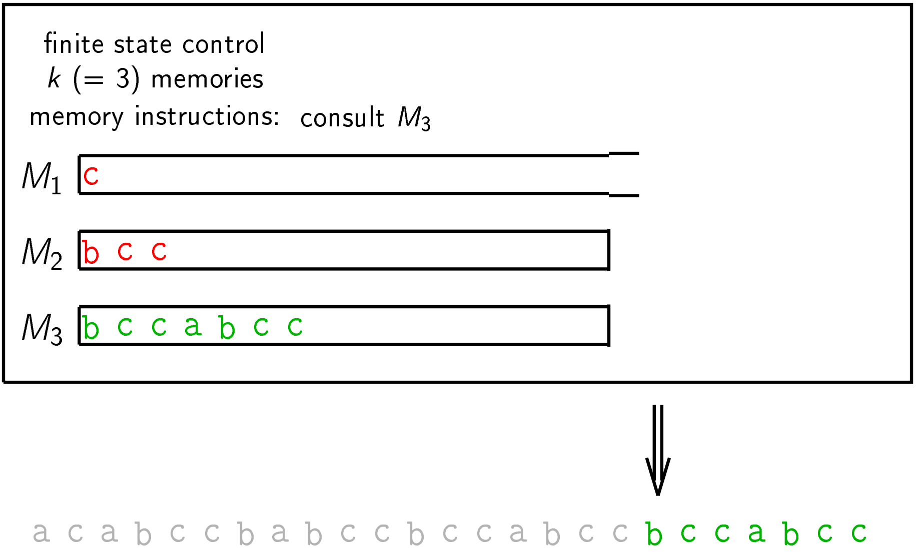

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k $(= 3)$ memories

memory instructions:

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M_2 b c c

M_3 b c c a b c c

$\Downarrow$

a c a b c c b a b c c b c c a b c c b c c a b c c

Memory Automata (MFA)

finite state control

k ($= 3$) memories

memory instructions: close M_1

M_1	c b c c a b c c
M_2	b c c
M_3	b c c a b c c

a c a b c c b a b c c b c c a b c c b c c a b c c

Determinism of MFA

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- $\mathcal{L}(\text{MFA}) = \mathcal{L}(\text{pseudo-det- MFA})$ (extended subset construction).

Determinism of MFA

- Pseudo deterministic: For every state, symbol b and memory i, at most one move that reads b and at most one move that consults memory i.
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- $\mathcal{L}(\text{MFA}) = \mathcal{L}(\text{pseudo-det- MFA})$ (extended subset construction).
- $\mathcal{L}(\text{DMFA}) \subset \mathcal{L}(\text{MFA})$ $(\{ww \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\} \notin \mathcal{L}(\text{DMFA}))$.

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Theorem

Every MFA(k) *can be transformed into a equivalent* MFA(k^2) *that is pseudo-deterministic and nested.*

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NFA reads $a \in \Sigma$	$\Rightarrow$	MFA reads $a \in \Sigma$
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Lemma

Let $M \in \mathrm{NFA}$ with $L(M) \subseteq \Sigma^{[*]}$. Then $\mathcal{D}(L(M)) = L(\psi_{\mathcal{D}}(M))$.

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Question

Given a regular expression r with $L(r) \in \Sigma^{[*]}$. Is it possible to transform r into r' with the REGEX property and $\mathcal{D}(L(r)) = \mathcal{D}(L(r'))$.

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Transform N into a regular expression r with $L(r) = L(N)$ that has the REGEX property.

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We have separated the "regular"-part from the "reduplication"-part.

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- Decision problems for ref-REG.
- Investigate ref-$\mathcal{L}$ for other language classes $\mathcal{L}$, e. g., ref-CF.

Thank you very much for your attention.