Pattern Matching with Variables: A Multivariate Complexity Analysis

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Words with Coloured Holes

A word with (coloured) *holes*. . .

$$\mathtt{a}\,\mathtt{b}\,\square\,\mathtt{c}\,\square\,\mathtt{c}\,\mathtt{b}\,\square\,\mathtt{b}\,\square\,\mathtt{c}\,\mathtt{a}\,\square$$

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$$\mathtt{a\,b}\,{\color{blue}\square}\,\mathtt{c}\,{\color{green}\square}\,\mathtt{c\,b}\,{\color{blue}\square}\,\mathtt{b}\,{\color{green}\square}\,\mathtt{c\,a}\,{\color{green}\square}$$

. . . can be repaired. . .

$$\begin{aligned} \mathtt{a\,b\,a} &\to {\color{blue}\square} \\ \mathtt{c\,b} &\to {\color{green}\square} \end{aligned}$$

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$$\mathtt{a\,b}\,\square\,\mathtt{c}\,\square\,\mathtt{c\,b}\,\square\,\mathtt{b}\,\square\,\mathtt{c\,a}\,\square$$

. . . can be repaired. . .

$$\mathtt{a\,b\,a} \to \square$$

$$\mathtt{c\,b} \to \square$$

. . . by filling in new words:

$$\mathtt{a\,b}\,\boxed{\mathtt{a\,b\,a}}\,\mathtt{c}\,\boxed{\mathtt{c\,b}}\,\mathtt{c\,b}\,\boxed{\mathtt{a\,b\,a}}\,\mathtt{b}\,\boxed{\mathtt{c\,b}}\,\mathtt{c\,a}\,\boxed{\mathtt{c\,b}}$$

A Special Kind of Pattern Matching

For given

α (a word with coloured holes),

w (a word without holes),

is it possible to fill the holes of α in such a way that we obtain w?

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$$\alpha = {\color{green}\square}\, \mathtt{a}\, \mathtt{a}\, {\color{blue}\square}\, {\color{green}\square}\, {\color{blue}\square}\, \mathtt{c}\, \mathtt{b}\, {\color{green}\square}$$

$$u = \mathtt{a\,c\,a\,a\,a\,b\,c\,b\,a\,a\,c\,a\,b\,c\,b\,a\,c\,b\,a\,c}$$

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$$\alpha = {\color{green}\square}\, \mathtt{a}\, \mathtt{a}\, {\color{blue}\square}\, {\color{green}\square}\, {\color{blue}\square}\, \mathtt{c}\, \mathtt{b}\, {\color{green}\square}$$

$$v = \mathtt{c}\, \mathtt{c}\, \mathtt{b}\, \mathtt{a}\, \mathtt{a}\, \mathtt{c}\, \mathtt{c}\, \mathtt{b}\, \mathtt{c}\, \mathtt{b}\, \mathtt{c}\, \mathtt{c}\, \mathtt{b}$$

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$$w = \mathtt{a\,b\,b\,a\,a\,b\,a\,b\,c\,a\,b\,c\,b}$$

Some Notations and Definitions

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$\alpha \in (\Sigma \cup X)^+$	is a pattern	$\alpha := x_1 \mathtt{a} x_2 x_1 \mathtt{ba} x_2 x_1 x_3$
$X \to \Sigma^+$	is a substitution	$h(x_1) := \mathtt{ab},\ h(x_2) := \mathtt{bcc}$

Pattern Matching with Variables

VPatMatch
Instance: A pattern $\alpha \in (\Sigma \cup X)^*$, *a word* $w \in \Sigma^*$.
Question: Does there exist a substitution h *with* $h(\alpha) = w$*?*

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Two variants:

E-VPatMatch	Substitution may map variables to the empty word ε.
NE-VPatMatch	Substitution can only map to non-empty words.

A Very Brief History

Three branches:

- Learning theory and Language theory (1980 - today):
  - Membership problem of Angluin's pattern languages.
  - First NE-case, later E-case.
  - Word equations, where one side is "variable-free".

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  - Baker's parameterised matching (finding repetitions in program code).
  - A. Amir, Y. Aumann, R. Cole, M. Lewenstein: function matching.
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  - Only NE-case.
- The "real" world (?? - today):
  - Matchtest for regular expressions with backreferences.
  - Nowadays a standard tool in text editors (grep, emacs, . . .) and programming language (Perl, Java, Python, . . .).

NP-Completeness

Theorem (Angluin 1980, Ehrenfeucht and Rozenberg 1979)

If $|\Sigma| \geq 2$*, then* E- *and* NE-$\mathrm{VP}\textsc{at}\mathrm{M}\textsc{atch}$ *are NP-complete.*

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3CNF formula (without negated variables)

$$\psi = (v_1 \vee v_2 \vee v_3) \wedge (v_2 \vee v_4 \vee v_5) \wedge (v_3 \vee v_1 \vee v_3) \wedge (v_4 \vee v_1 \vee v_2)$$

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$\mathrm{E\text{-}VP{\scriptsize AT}M{\scriptsize ATCH}}$ instance:

$$\alpha_\psi = x_1 x_2 x_3 \, \mathtt{b} \, x_2 x_4 x_5 \, \mathtt{b} \, x_3 x_1 x_3 \, \mathtt{b} \, x_4 x_1 x_2$$
$$w_\psi = \mathtt{a}\,\mathtt{b}\,\mathtt{a}\,\mathtt{b}\,\mathtt{a}\,\mathtt{b}\,\mathtt{a}.$$

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E-VPatMatch instance:

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$$w_\psi = \mathtt{a}\,\mathtt{b}\,\mathtt{a}\,\mathtt{b}\,\mathtt{a}\,\mathtt{b}\,\mathtt{a}.$$

$\exists\, h : h(\alpha_\psi) = w_\psi$ iff ψ is "1-in-3-satisfiable".

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- we are only interested in texts of size at most 50,
- we are only interested in injective substitutions,
- in our patterns every variable occurs at most twice,
- we are only interested in patterns without any terminal symbols and we only consider substitutions of the form $h : X \to \{\mathtt{a}, \mathtt{b}, \varepsilon\}$?
  (i. e., for some u over some alphabet Γ and some $w \in \{\mathtt{a}, \mathtt{b}\}^*$, can we obtain w by replacing every $x \in \Gamma$ in u by either $\mathtt{a}$ or $\mathtt{b}$ or deleting it?)

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$|\alpha|_x$ is the number of Occ. of x in α $\qquad |\alpha|_{x_1} = 3$

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Types of VPatMatch:

- Substitutions can be erasing or must be non-erasing.

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2^3 types, 2^5 combinations of parameters $\rightarrow$ 256 versions of $\textsc{VPatMatch}$.

Research Questions

256 Questions of the following form:

Main Research Question

For any type X of $\textsc{VPatMatch}$ and for any subset P of parameters, can we bound the parameters in P by constants, such that type X of $\textsc{VPatMatch}$ is still NP-complete?

First Observations

Theorem (Geilke, Zilles, 2011)

If

$$\begin{aligned} |\mathrm{var}(\alpha)| &\leq c \text{ or} \\ |w| &\leq c, \end{aligned}$$

for some constant c, then all variants of $\textsc{VPatMatch}$ *are in P.*

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Observation

If

$$\begin{aligned} |\alpha|_x &= 1 \text{ or} \\ |\Sigma| &= 1, \end{aligned}$$

then all variants of $\textsc{VPatMatch}$ are in P.

The Non-injective Case

Theorem

Erasing, non-injective VPatMatch *is NP-complete,*

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Erasing, non-injective VPATMATCH *is NP-complete,*

- *even if*

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- *even if terminal-free and*

$$\begin{aligned} |h(x)| &\leq 1\,, \\ |\alpha|_x &\leq 8, \\ |\Sigma| &\leq 2\,. \end{aligned}$$

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Theorem

(Non-)Erasing, non-injective $\textsc{VPatMatch}$ *is NP-complete,*

- *even if*

$$\begin{aligned} |h(x)| &\leq 1\,(3), \\ |\alpha|_x &\leq 2\,(2), \\ |\Sigma| &\leq 2\,(2). \end{aligned}$$

- *even if terminal-free and*

$$\begin{aligned} |h(x)| &\leq 1\,(3), \\ |\alpha|_x &\leq \cancel{8}\,2\,(3), \\ |\Sigma| &\leq 2\,(4). \end{aligned}$$

The Injective Case 1/2

Theorem

Let $c_1, c_2 \in \mathbb{N}$. All injective variants of $\textsc{VPatMatch}$, *restricted to*

$$\begin{aligned} |h(x)| &\leq c_1, \\ |\Sigma| &\leq c_2, \end{aligned}$$

are in P.

The Injective Case 2/2

For all other injective variants, we have NP-completeness, but the constants are a bit larger.

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Theorem

Injective, erasing or non-erasing, terminal-free or non-terminal-free $\textsc{VPatMatch}$ *is NP-complete,*

- *even if*

$$\begin{aligned} |h(x)| &\leq 19, \\ |\alpha|_x &\leq 4, \end{aligned}$$

- *even if*

$$\begin{aligned} |\alpha|_x &\leq 9, \\ |\Sigma| &\leq 5. \end{aligned}$$

Further Research 1/2

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Dichotomy Result

For any variant X of $\textsc{VPatMatch}$, for any subset P of parameters and for any set C of specific bounds for the parameters in P, is the variant X of $\textsc{VPatMatch}$ still NP-complete if the parameters of P are bounded by the constants in C?

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For any variant X of $\mathrm{VP{\scriptsize AT}M{\scriptsize ATCH}}$ and for any subset P of parameters, can we bound the parameters in P by constants, such that variant X of $\mathrm{VP{\scriptsize AT}M{\scriptsize ATCH}}$ is still NP-complete?

Dichotomy Result for Erasing and Non-injective Case

Let $c_1, c_2, c_3 \in \mathbb{N}$. Erasing, non-injective $\mathrm{VP{\scriptsize AT}M{\scriptsize ATCH}}$, restricted to

$$\begin{aligned} |h(x)| &\leq c_1, \\ |\alpha|_x &\leq c_2, \\ |\Sigma| &\leq c_3, \end{aligned}$$

is NP-Complete **if and only if** $c_1 \geq 1$, $c_2 \geq 2$, $c_3 \geq 2$.

Further Research 2/2

Parameterized Complexity

Consider the parameters ($|\operatorname{var}|$, $|\Sigma|$, $\ldots$) as parameters in terms of *parameterized complexity theory* and investigate the parameterized complexity of the corresponding parameterized problems.

Thank you very much for your attention.