Inside the Class of REGEX Languages

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$\{\mathtt{a}^n\mathtt{b}\mathtt{a}^n\mathtt{c}\mathtt{a}^n\mathtt{b}\mathtt{a}^n \mid n \geq 0\}$

Practical Relevance of REGEX

- REGEX are intensely applied in practice...
  - Traditional and Modern `grep`
  - `vi`
  - Modern `sed`
  - GNU `Emacs`
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- ...even though their membership problem is NP-complete.

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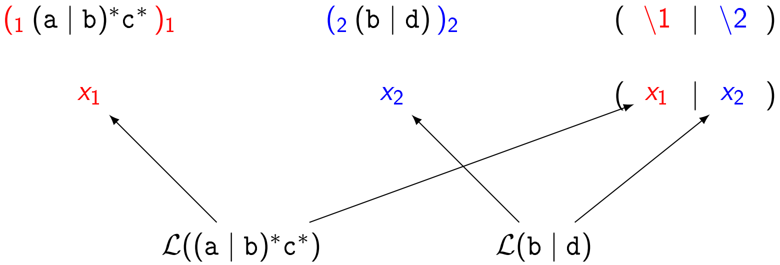

A More Complex Example

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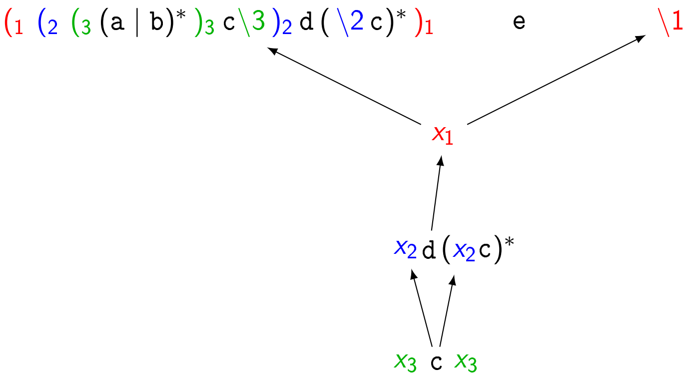

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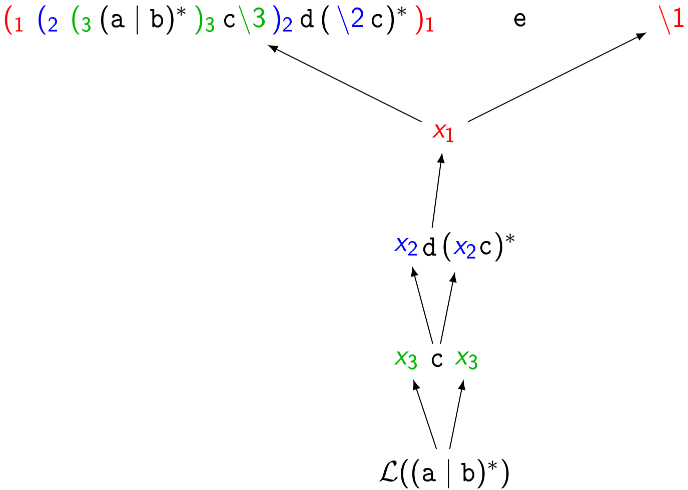

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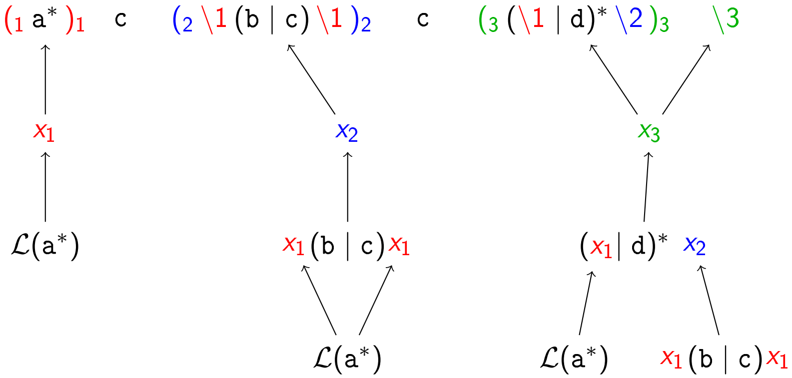

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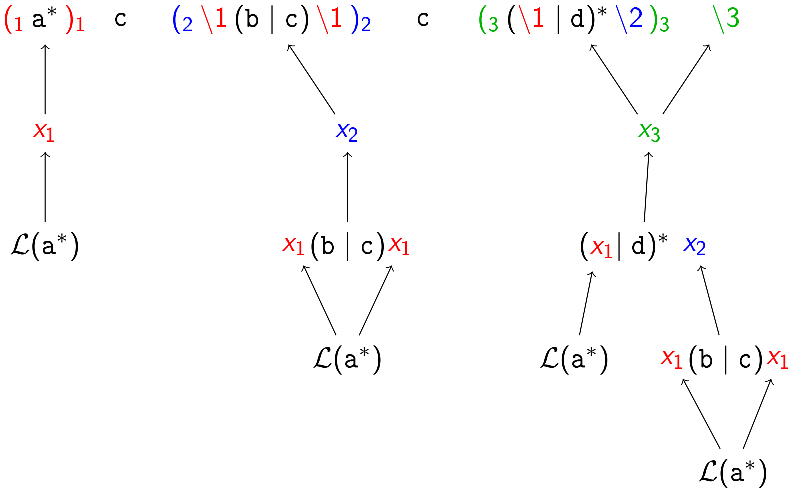

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- Our approach: Study REGEX by investigating alternative ways to combine regular expressions and homomorphic replacement...
- ...without exceeding the expressive power of REGEX languages.
- Informally: Take regular expressions, take some mechanism of homomorphic replacement, combine them and see how much of the class of REGEX languages we actually get.

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For any language class $\mathfrak{L}$,
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Proposition

$\mathcal{L}_{\mathsf{REG}}(\mathsf{PAT}) \subseteq \mathcal{L}(\mathsf{REGEX})$.

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Hence, the aspect of regular expressions cannot be limited to the type languages.

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$\{(\mathtt{b}^n\mathtt{c})^m \mid n \geq 0, m \geq 1\}$.

Expressive Power

Theorem

$\mathcal{L}_{\{\Sigma^*\}}(\mathsf{PAT}) \subset \mathcal{L}_{\mathsf{REG}}(\mathsf{PAT}) \subset \mathcal{L}_{\mathsf{REG}}(\mathsf{PAT}_{\mathsf{ro}})$.

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Example: $q := (x_1 \to \mathtt{a}^*, x_2 \to x_1(\mathtt{c} \mid \mathtt{d})x_1, x_3 \to x_1 \mathtt{c} x_2)$.

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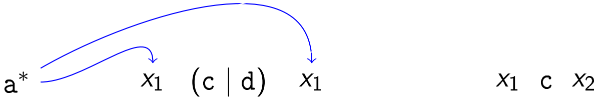

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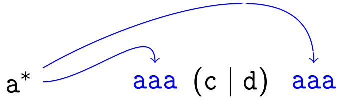

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$\mathtt{a}^*$ $\mathtt{aaa}$ $(\mathtt{c} \mid \mathtt{d})$ $\mathtt{aaa}$ x_1 $\mathtt{c}$ $\mathtt{aaacaaa}$

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$\mathcal{L}_{\text{it}}(q) = \{\mathtt{a}^k \mathtt{c} \mathtt{a}^m u \mathtt{a}^m \mid k, m \in \mathbb{N}_0, u \in \{\mathtt{c}, \mathtt{d}\}\}$ is the *language generated by q with respect to iterated substitution*.

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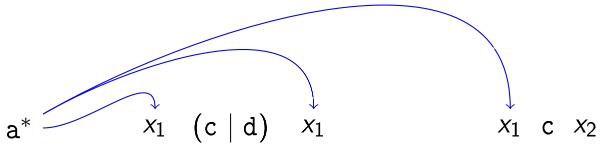

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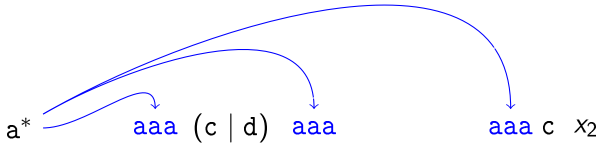

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$\texttt{a}^*$ aaa (c | d) aaa aaa c aaadaaa

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Pattern Expression Languages

Proposition

[Campeanu and Yu] For every $p \in \mathrm{PE}$, $\mathcal{L}_{\mathrm{it}}(p)$ is a REGEX language.

Theorem

$\mathfrak{L}_{\mathrm{ro},\infty} = \mathcal{L}_{\mathrm{it}}(\mathrm{PE})$.

Iterated vs. Uniform Substitution

Proposition

Let $p := (x_1 \rightarrow r_1, \ldots, x_m \rightarrow r_m) \in \mathsf{PE}$.

- $\mathcal{L}_{\text{uni}}(p) \subseteq \mathcal{L}_{\text{it}}(p)$,
- if, for every i, j, $1 \leq i < j \leq m$, $\text{var}(r_i) \cap \text{var}(r_j) = \emptyset$, then $\mathcal{L}_{\text{it}}(p) \subseteq \mathcal{L}_{\text{uni}}(p)$.

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Theorem

$\mathcal{L}_{\mathrm{it}}(\mathrm{PE}) \subset \mathcal{L}_{\mathrm{uni}}(\mathrm{PE})$.

Notation

A REGEX r is *star-free initialised* iff every referenced subexpression does not occur under a star.

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A REGEX r is *star-free initialised* iff every referenced subexpression does not occur under a star.

- $((_1(\mathtt{a} \mid \mathtt{b})^*)_1\mathtt{b}\backslash 1)^*\mathtt{b}\backslash 1$
- $(_1(\mathtt{a} \mid \mathtt{b})^*)_1\backslash 1(_2\mathtt{c}^*)_2(\mathtt{d}\backslash 1\backslash 2)^*$

PE w. r. t. uniform subst. vs. star-free initialised REGEX

Lemma

For every pattern expression p, there exists a star-free initialised REGEX r *with* $\mathcal{L}_{\mathsf{uni}}(p) = \mathcal{L}(r)$.

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For every star-free initialised REGEX r*, there exists a pattern expression* p *with* $\mathcal{L}(r) = \mathcal{L}_{\text{uni}}(p)$.

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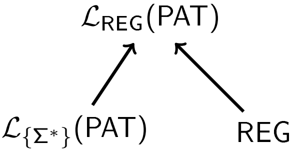
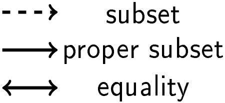
Theorem

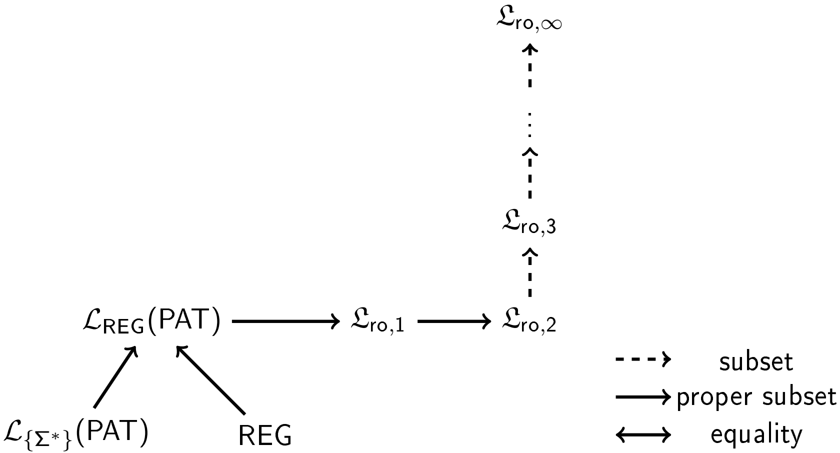
$\mathcal{L}(\text{REGEX}_{\text{sfi}}) = \mathcal{L}_{\text{uni}}(\text{PE})$.

$\mathcal{L}_{\{\Sigma^*\}}(\text{PAT})$ REG

- - -> subset
⟶ proper subset
⟷ equality


$\mathcal{L}_{\mathsf{REG}}(\mathsf{PAT})$
$\mathcal{L}_{\{\Sigma^*\}}(\mathsf{PAT})$
REG


subset
proper subset
equality


$\mathfrak{L}_{\text{ro},\infty}$
$\mathfrak{L}_{\text{ro},3}$
$\mathcal{L}_{\text{REG}}(\text{PAT})$
$\mathfrak{L}_{\text{ro},1}$
$\mathfrak{L}_{\text{ro},2}$
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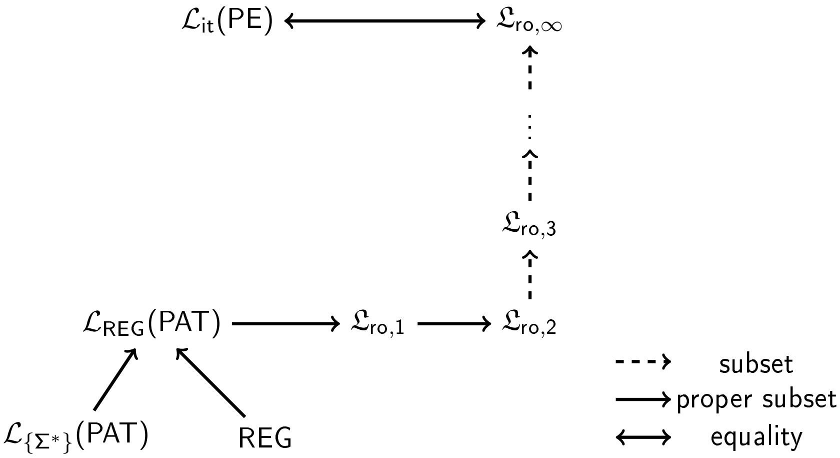
$\mathcal{L}_{it}(PE)$
$\mathfrak{L}_{ro,\infty}$
$\mathfrak{L}_{ro,3}$
$\mathcal{L}_{REG}(PAT)$
$\mathfrak{L}_{ro,1}$
$\mathfrak{L}_{ro,2}$
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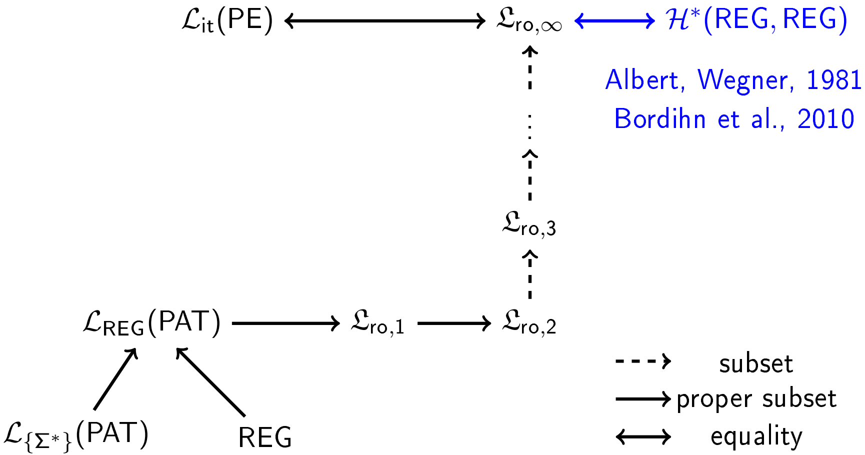
proper subset
equality


$\mathcal{L}_{it}(PE)$
$\mathfrak{L}_{ro,\infty}$
$\mathcal{H}^*(REG, REG)$
Albert, Wegner, 1981
Bordihn et al., 2010
$\mathfrak{L}_{ro,3}$
$\mathcal{L}_{REG}(PAT)$
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$\mathcal{L}_{\{\Sigma^*\}}(PAT)$
REG
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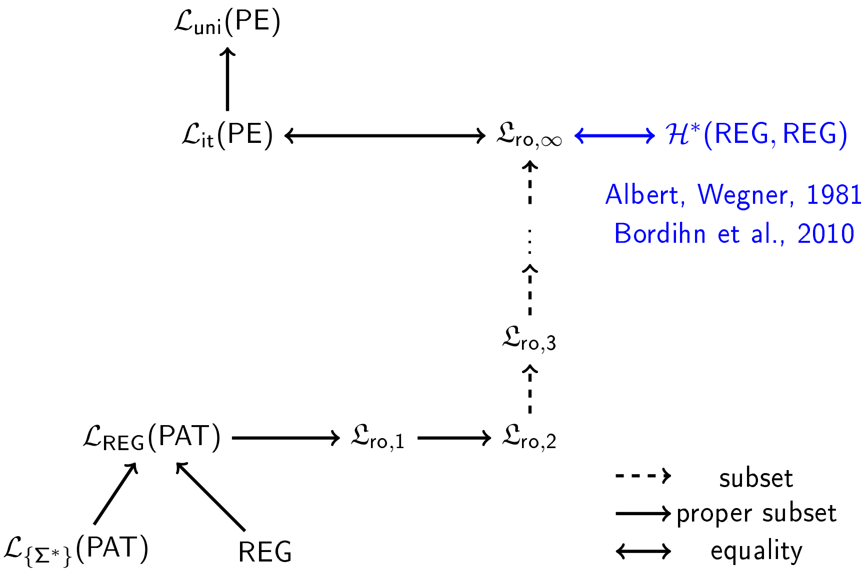
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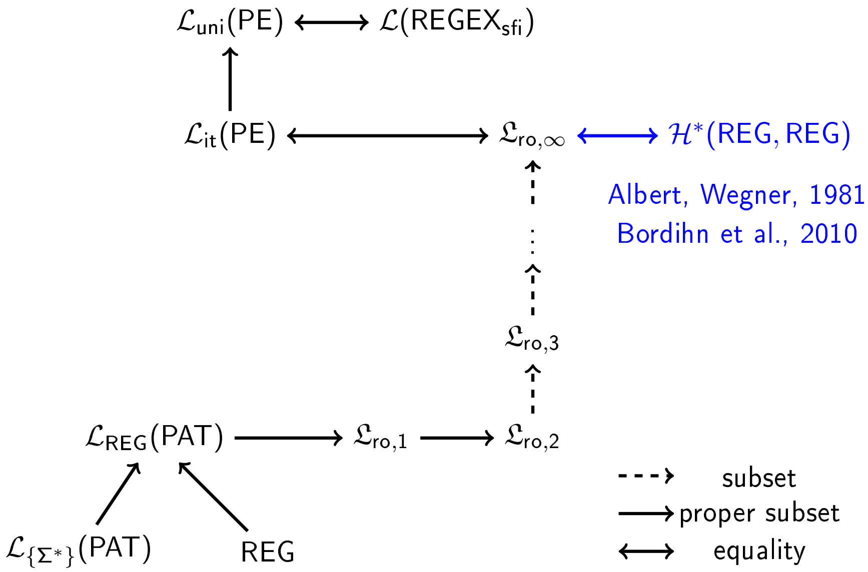
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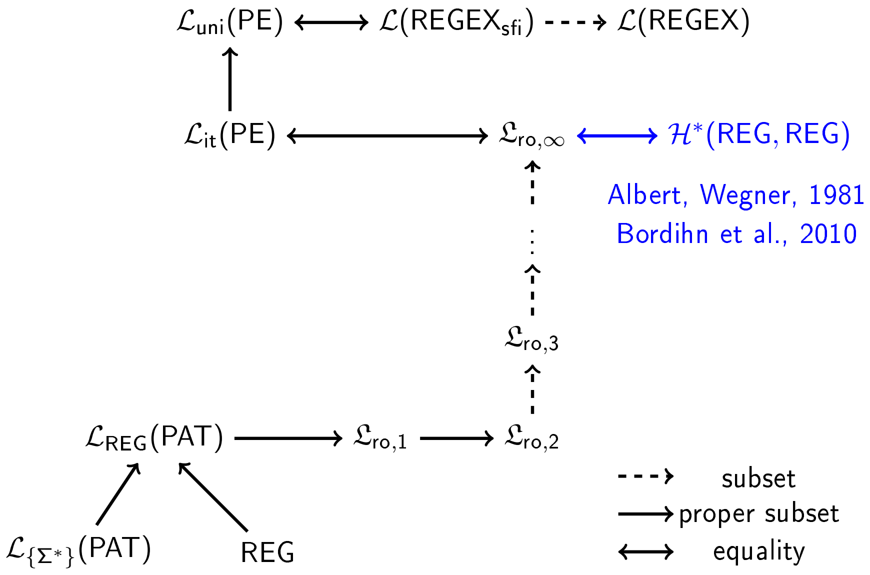
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