Patterns with Bounded Treewidth

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Pattern languages

$x_1\ \mathtt{aa}\ x_2\ x_1\ x_2\ \mathtt{cb}\ x_1$

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$\{w \mid w = u\,\mathrm{aa}\,v\,u\,v\,\mathrm{cb}\,u, \text{ where } u, v \in \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}^*\}.$

The membership problem for pattern languages

Theorem

The membership problem for pattern languages is NP-complete (Angluin, 80).

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Main research task: Find classes of pattern languages with a polynomial membership problem.

Previous results

- Brute-force algorithm with runtime $O(|w|^k)$, where k is the number of variables.
- A pattern language is a regular language if
  - it is unary or
  - no variable in the pattern is repeated.
- $|w| \leq k$ for a constant k (Geilke, Zilles, ALT 2011)
- *Non-cross* patterns (e. g., $x_1 x_1 x_1 x_2 x_2 x_3 x_3 x_3$) (Shinohara, 82).
- Patterns with bounded *variable distance* (R., S., CIAA 2010).

Relational Structures

Definition (Relational structure)

A relational vocabulary τ is a finite set of relation symbols. Every relation symbol $R \in \tau$ has an arity $\mathrm{ar}(R) \geq 1$. A τ-structure $\mathcal{A}$ comprises a finite set A called the universe and, for every $R \in \tau$, an interpretation $R^{\mathcal{A}} \subseteq A^{ar(R)}$.

Example

Every graph can be interpreted as a relational structure $\mathcal{G}$, where the universe V is the set of vertices and the edges are given as an interpretation of a binary relation E.

We only consider relations with arity of at most 2.

Homomorphism problem for relational structures

Definition (Homomorphism)

Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures with universes A and B, respectively. A *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ is a mapping $h : A \to B$ such that for all $R \in \tau$ and for all $a_1, a_2, \ldots, a_{\mathrm{ar}(R)} \in A$, $(a_1, a_2, \ldots, a_{\mathrm{ar}(R)}) \in R^{\mathcal{A}}$ implies $(h(a_1), h(a_2), \ldots, h(a_{\mathrm{ar}(R)})) \in R^{\mathcal{B}}$.

Definition (Homomorphism problem)

The *homomorphism problem HOM* is the problem of deciding, for any structures $\mathcal{A}$ and $\mathcal{B}$, whether there exists a homomorphism from $\mathcal{A}$ to $\mathcal{B}$. For any set of structures C, by HOM(C) we denote the homomorphism problem, where the left hand structure is restricted to be from C.

Homomorphism problem for relational structures

Theorem

The homomorphism problem for relational structures is NP-complete.

α-structures

$$\alpha := x_1 \quad x_2 \quad x_1 \quad x_3 \quad x_2 \quad x_3 \quad \mathtt{b} \quad x_1 \quad x_2 \quad x_1$$

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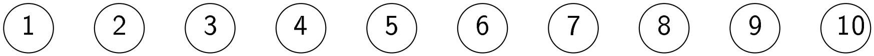

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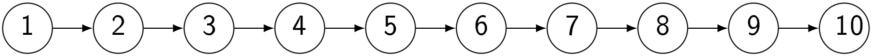

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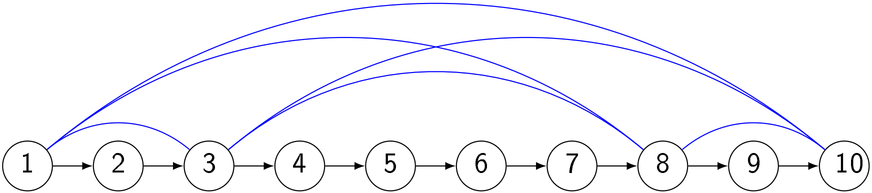

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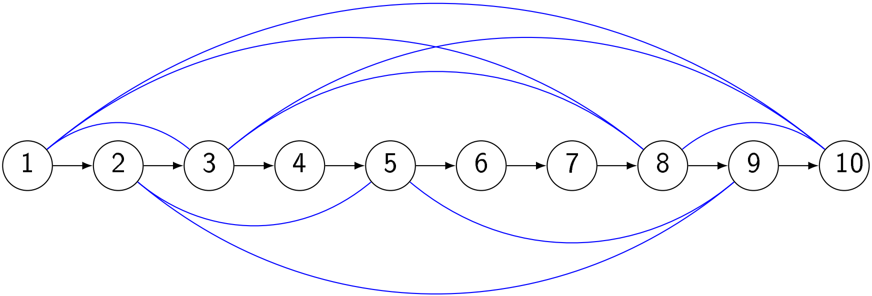

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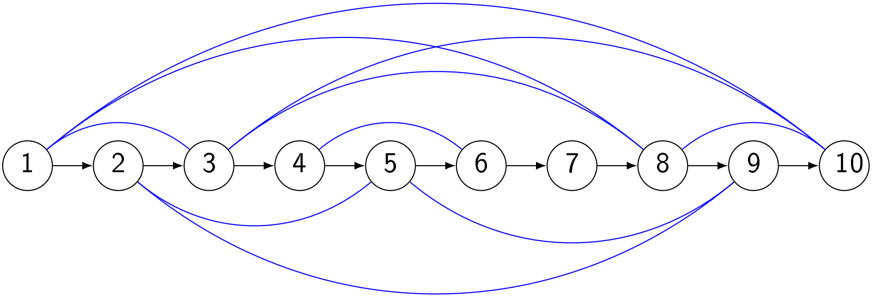

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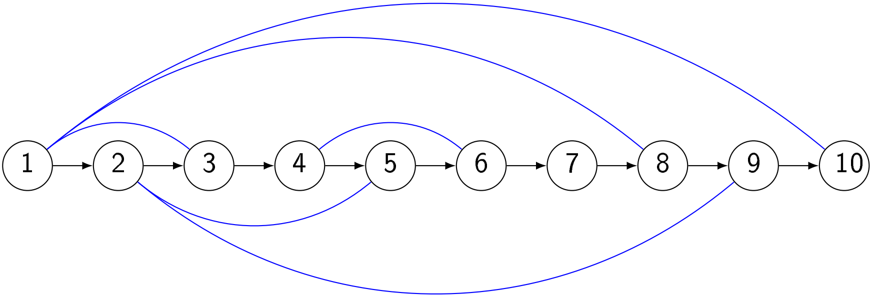

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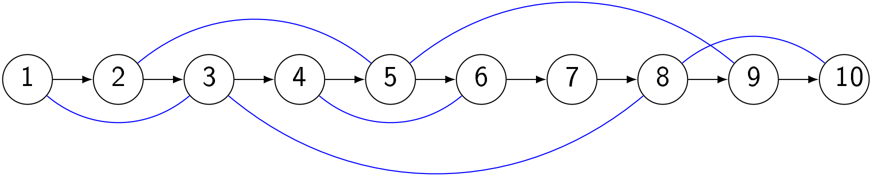

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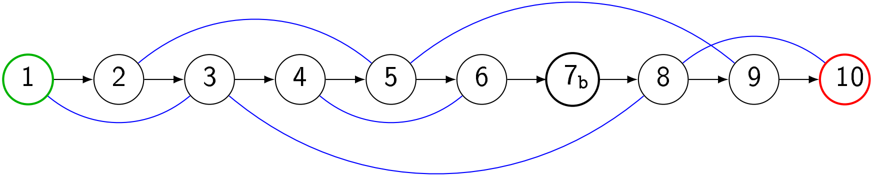

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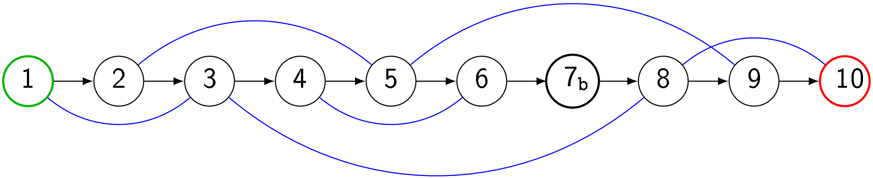

We denote α-structures by $\mathcal{A}_\alpha$ and the standard α-structure by $\mathcal{A}^s_\alpha$.

w-structures

$w := \mathtt{abab}$

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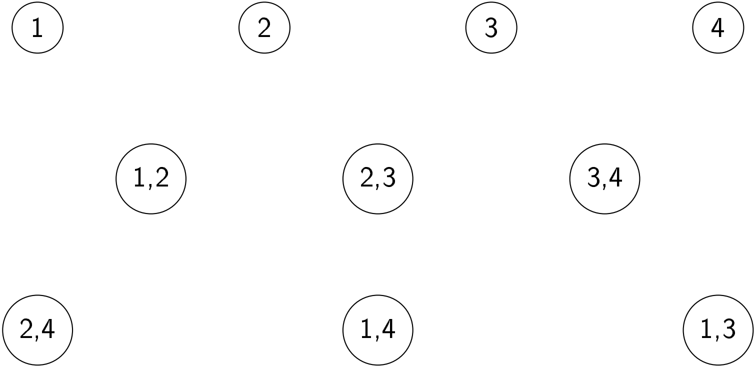

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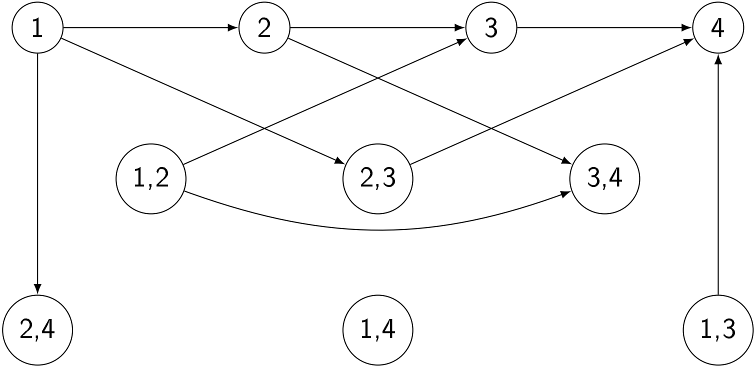

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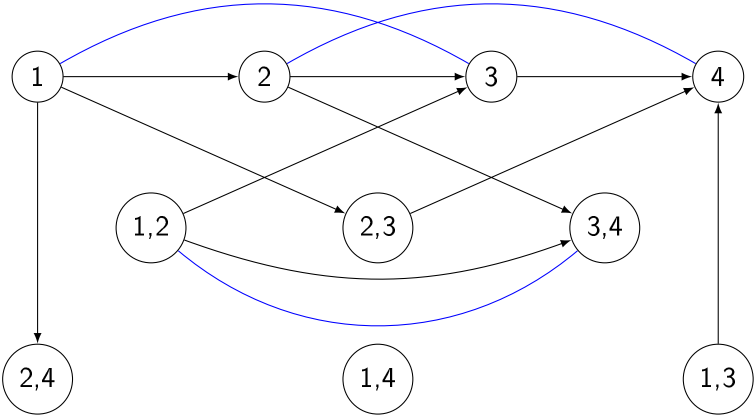

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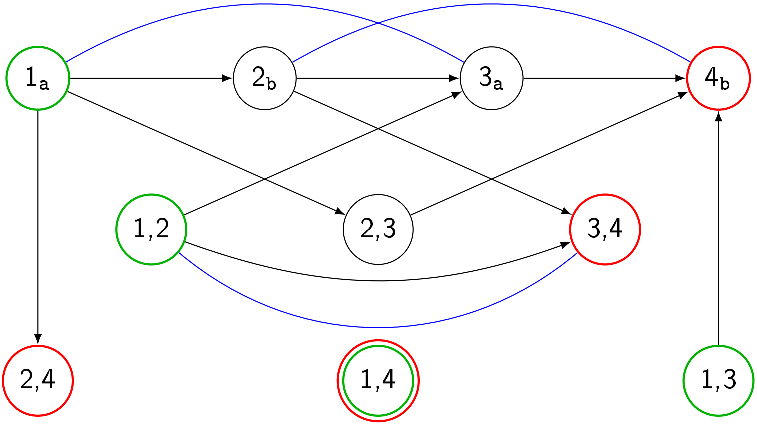

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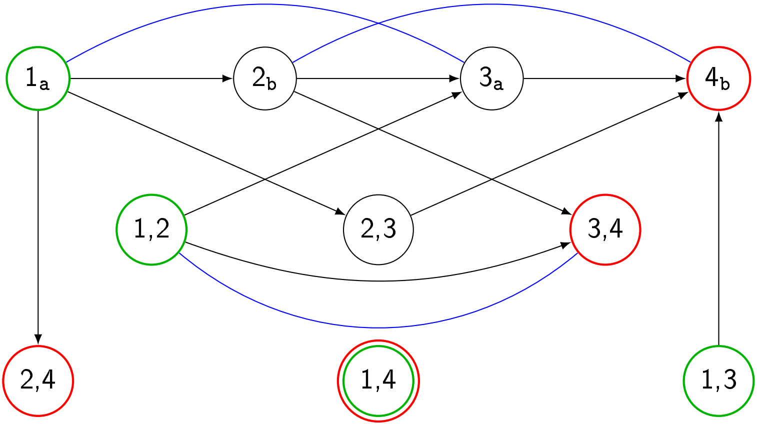

We denote w-structures by $\mathcal{A}_w$.

Membership problem $\leq$ homomorphism problem

Lemma

A word w is in the pattern language of α if and only if there exists a homomorphism from $\mathcal{A}_\alpha$ to $\mathcal{A}_w$, where $\mathcal{A}_\alpha$ is some α-structure.

A meta theorem

Theorem (Freuder, 90)

Let C be a class of relational structures with bounded treewidth. Then HOM(C) can be solved in polynomial time.

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Theorem (Meta theorem)

Let P be a class of patterns and let f be a polynomial time computable function that maps every $\alpha \in P$ to an α-structure. If, for some constant k, $\max\{\text{tw}(f(\alpha)) \mid \alpha \in P\} \leq k$, then the membership problem for P is decidable in polynomial time.

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Research task: Find classes of patterns with bounded treewidth.

Scope coincidence degree

Let α be a pattern.

- For every $y \in \text{var}(\alpha)$, the *scope of* y *in* α is defined by $\text{sc}_\alpha(y) := \{i, i+1, \ldots, j\}$, where i is the leftmost and j the rightmost position of y in α.
- The scopes of $y_1, y_2, \ldots, y_k \in \text{var}(\alpha)$ *coincide in* α if and only if $\bigcap_{1 \leq i \leq k} \text{sc}_\alpha(y_i) \neq \emptyset$.
- The *scope coincidence degree* of α ($\text{scd}(\alpha)$) is the maximum number of variables in α such that their scopes coincide.

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$$\alpha_2 := x_1 \; x_2 \; x_1 \; x_1 \; x_2 \; x_3 \; x_2 \; x_3 \; x_3$$

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$$\alpha_1 := x_1 \; x_2 \; x_1 \; x_3 \; x_2 \; x_3 \; x_1 \; x_2 \; x_3 \qquad \text{scd}(\alpha_1) = 3$$

$$\alpha_2 := x_1 \; x_2 \; x_1 \; x_1 \; x_2 \; x_3 \; x_2 \; x_3 \; x_3 \qquad \text{scd}(\alpha_2) = 2$$

Scope coincidence degree

Lemma

Let α *be a pattern. Then* $\text{tw}(\mathcal{A}^s_\alpha) \leq \text{scd}(\alpha) + 1$.

Theorem

Let $k \in \mathbb{N}$ *and* $P := \{\alpha \mid \text{scd}(\alpha) \leq k\}$. *The membership problem for the class* P *is decidable in polynomial time.*

Mildly entwined patterns

Let α be a pattern.

- Two variables $x, y \in \text{var}(\alpha)$ are *entwined* iff $\alpha = \beta \cdot x \cdot \gamma_1 \cdot y \cdot \gamma_2 \cdot x \cdot \gamma_3 \cdot y \cdot \delta$.
- If no two variables are entwined, then α is *nested*.
- α is *closely entwined* iff $\alpha = \beta \cdot x \cdot \gamma_1 \cdot y \cdot \gamma_2 \cdot x \cdot \gamma_3 \cdot y \cdot \delta$ with $|\gamma_2|_x = |\gamma_2|_y = 0$ implies $\gamma_2 = \varepsilon$.
- α is *mildly entwined* iff it is closely entwined and, for every $x \in \text{var}(\alpha)$, if $\alpha = \beta \cdot x \cdot \gamma \cdot x \cdot \delta$ with $|\gamma|_x = 0$, then γ is nested.

Mildly entwined patterns

$$\alpha = \quad x_1 \; x_3 \; x_4 \; x_4 \; x_3 \; x_3 \; x_1 \; x_2 \; x_3 \; x_5 \; x_5 \; x_2 \; x_5 \; x_6 \; x_6 \; x_2$$

Mildly entwined patterns

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$\Rightarrow \alpha$ is mildly entwined.

Mildly entwined patterns

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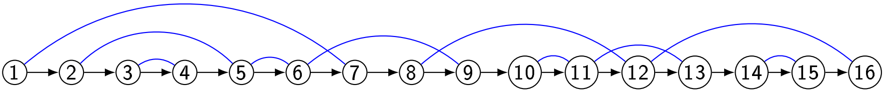

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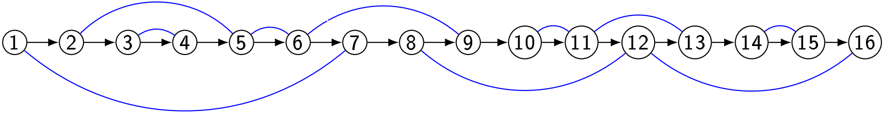

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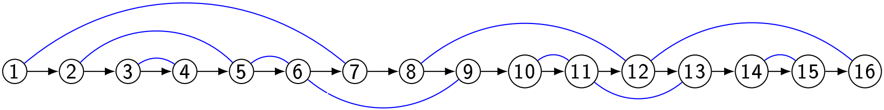

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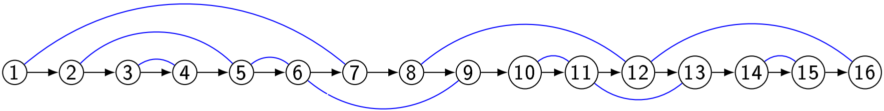

Definition

A graph is *outerplanar* iff it can be drawn in a planar way such that no vertex is entirely surrounded by edges (or, equivalently, all vertices lie on the exterior face).

Mildly entwined patterns

Lemma

A pattern α is mildly entwined if and only if $\mathcal{A}^s_\alpha$ is outerplanar.

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Theorem (Bodlaender 86)

If $\mathcal{G}$ is an outerplanar graph, then $\mathrm{tw}(\mathcal{G}) \leq 2$.

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Theorem

Let $P := \{\alpha \mid \alpha$ is mildly entwined$\}$. The membership problem for the class P is decidable in polynomial time.

Mildly entwined vs. bounded scope coincidence degree

Observation

For every k, $k \geq 2$, $\{\alpha \mid \mathsf{scd}(\alpha) \leq k\} \# \{\alpha \mid \alpha \text{ is mildly entwined}\}$.