Finding Shuffle Words that Represent Optimal Scheduling of Shared Memory Access

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Motivation

- Extended regular expressions (and pattern languages (introduced by Angluin in 1980))
- NP-complete membership problem
- Nontrivial subclasses with polynomial membership problem
- Automata as general algorithmic frameworks

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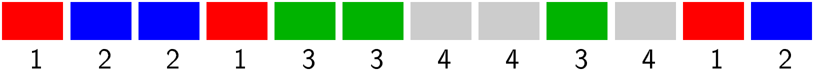

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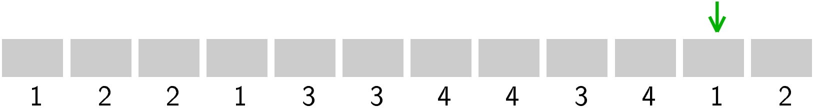

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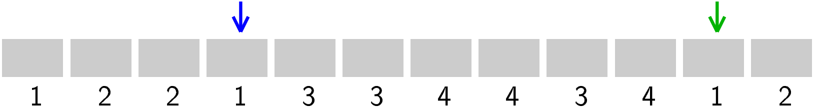

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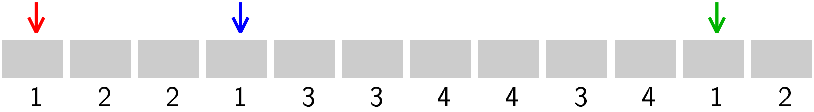

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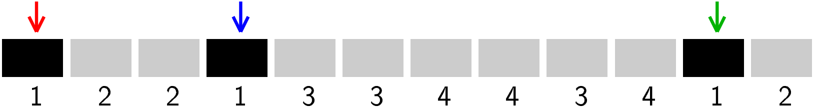

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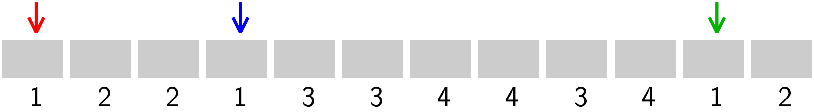

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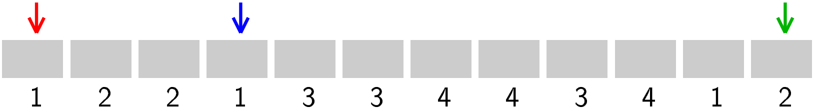

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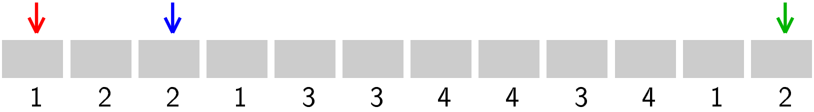

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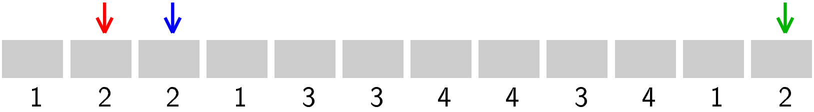

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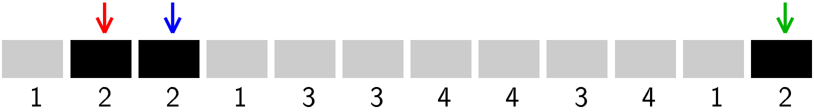

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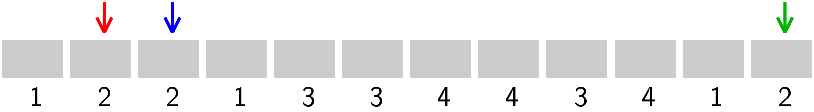

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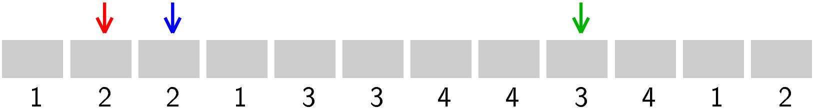

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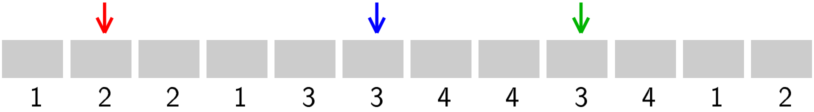

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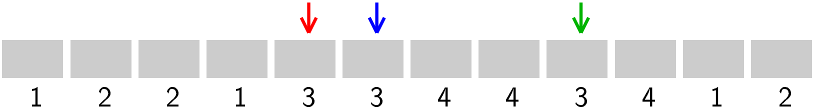

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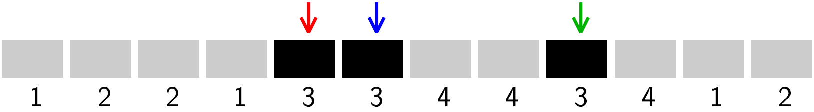

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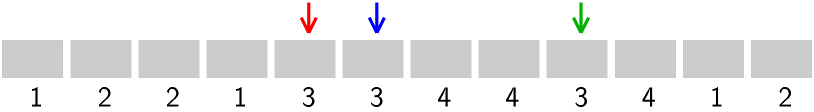

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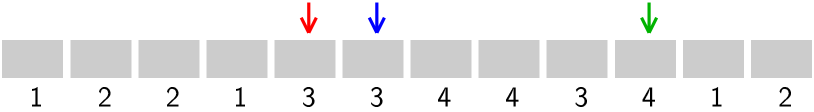

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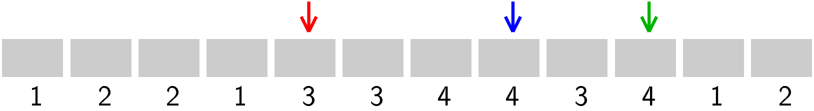

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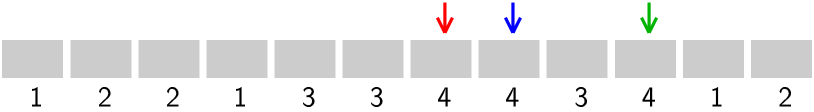

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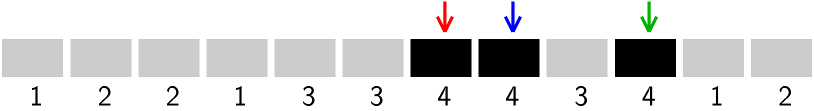

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Scheduling example

Process 1: a b a c b c

Memory: ☐ ☐

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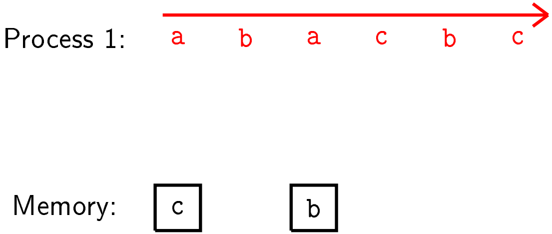

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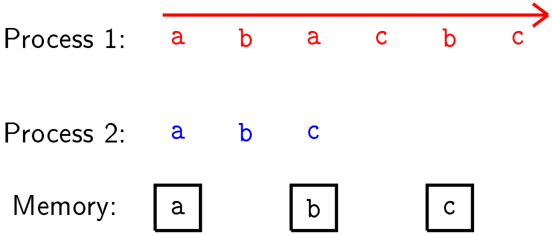

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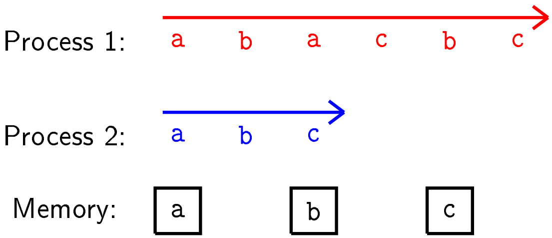

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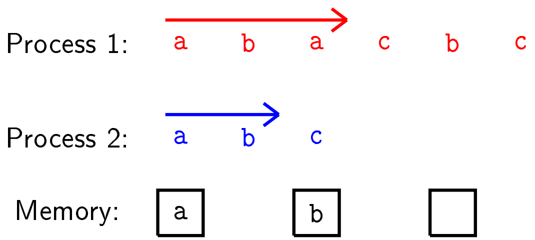

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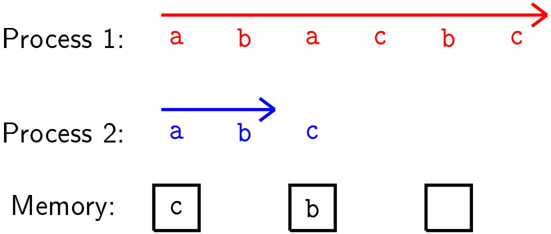

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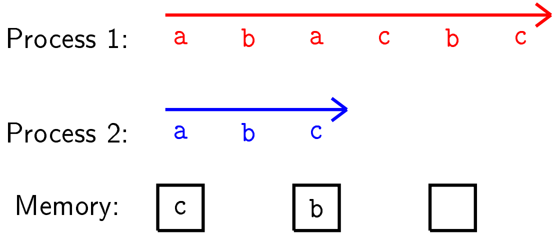

The Scope Coincidence Degree

- The *scope* of symbol a coincides with the scope of symbol b in $w \in \Sigma^*$ iff $w = \cdots \mathtt{a} \cdots \mathtt{b} \cdots \mathtt{a} \cdots$ or $w = \cdots \mathtt{b} \cdots \mathtt{a} \cdots \mathtt{b} \cdots$.
- Let $w \in \Sigma^*$. $\mathsf{scd}(w)$ denotes the maximum number of symbols in w, the scopes of which are pairwise coinciding.

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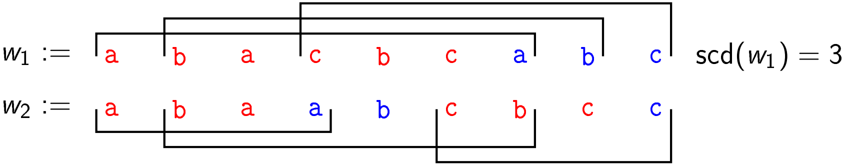

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Generalised Problem on Shuffle Words

SWminSCD_Σ

INPUT: $(w_1, \ldots, w_k) \in (\Sigma^+)^k$, for some $k \in \mathbb{N}$.
OUTPUT: $w \in w_1 \sqcup \ldots \sqcup w_k$ such that there exists no $w' \in w_1 \sqcup \ldots \sqcup w_k$ with $\text{scd}(w') < \text{scd}(w)$.

$w_1 \sqcup w_2$ denotes the *shuffle* of $w_1, w_2 \in \Sigma^*$.

Naive approach to SWminSCD_Σ

Let $w_1, w_2, \ldots, w_k \in \Sigma^*$, where $n := |w_1 \cdot w_2 \cdot \ldots \cdot w_k|$.

$$|w_1 \sqcup w_2 \sqcup \ldots \sqcup w_k| \leq \binom{n}{|w_1|,\ldots,|w_k|} = \frac{n!}{|w_1|! \times \ldots \times |w_k|!} \,.$$

Computing Shuffle Words Step by Step

Let $w_1, w_2, \ldots, w_k \in \Sigma^*$ be an input $\mathrm{SWminSCD}_\Sigma$.

- $u_1 := \mathrm{SWminSCD}_\Sigma(w_1, w_2)$,
- $u_2 := \mathrm{SWminSCD}_\Sigma(u_1, w_3)$,
- $u_3 := \mathrm{SWminSCD}_\Sigma(u_2, w_4)$,
- $\ldots$

Computing Shuffle Words Step by Step

Let $w_1 := \mathtt{aa}$, $w_2 := \mathtt{bb}$ and $w_3 := \mathtt{ba}$.

- $\mathtt{aabb} \in w_1 \sqcup w_2$ is optimal.
- For every $u \in \mathtt{aabb} \sqcup w_3$, $\mathsf{scd}(u) = 1$.
- $\mathtt{bbbaaa} \in w_1 \sqcup w_2 \sqcup w_3$ with $\mathsf{scd}(\mathtt{bbbaaa}) = 0$.

The Intrinsic Complexity

Let w_1, w_2, $w_3 \in \Sigma^*$ be arbitrary words.

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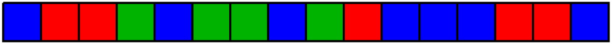

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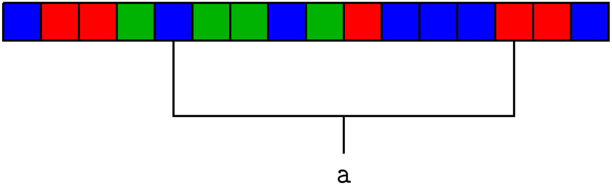

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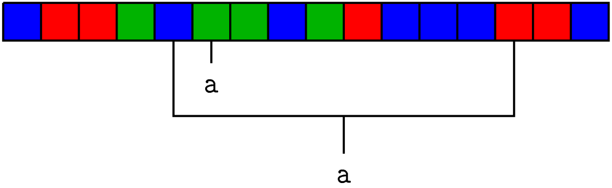

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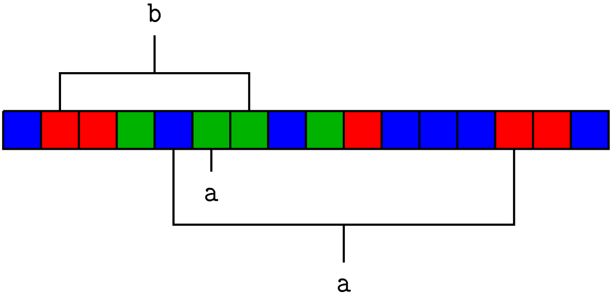

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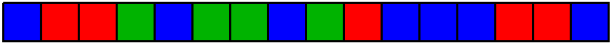

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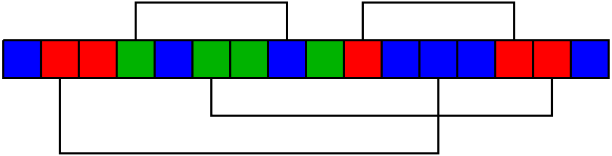

Our Algorithm

Main Result

SWminSCD_Σ on input $(w_1, w_2, \ldots, w_k)$ can be solved in time $O(|w_1 \cdot w_2 \cdot \ldots \cdot w_k| \times k^{|\Sigma|})$.

Our Algorithm

The idea of the algorithm:

Our Algorithm

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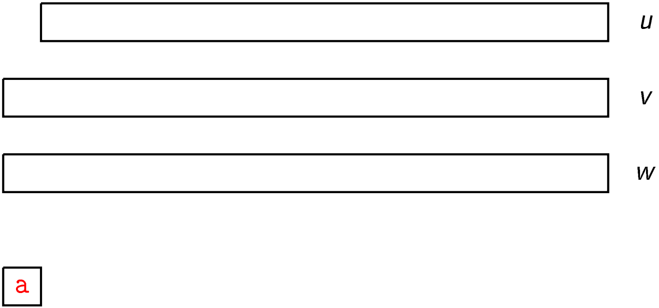

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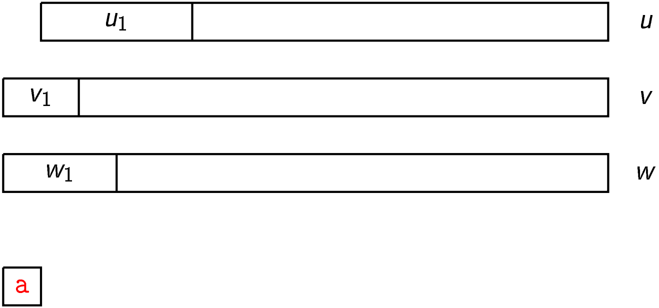

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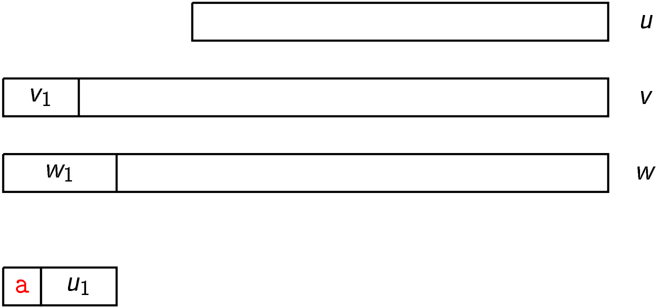

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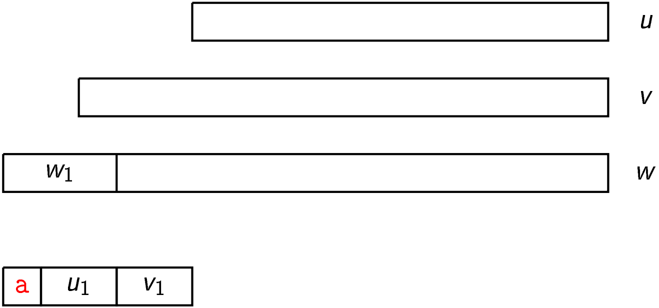

Our Algorithm

The idea of the algorithm:

a	u_1	v_1	w_1

Our Algorithm

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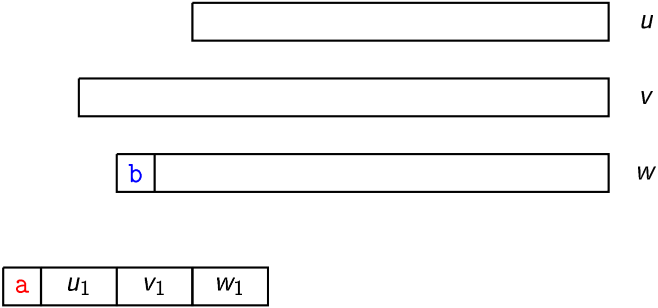

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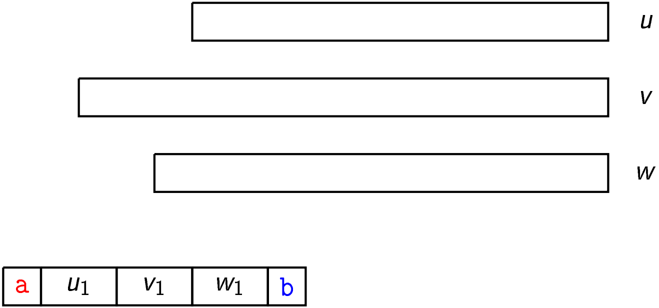

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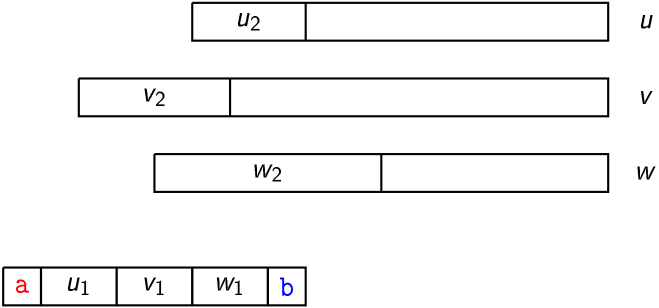

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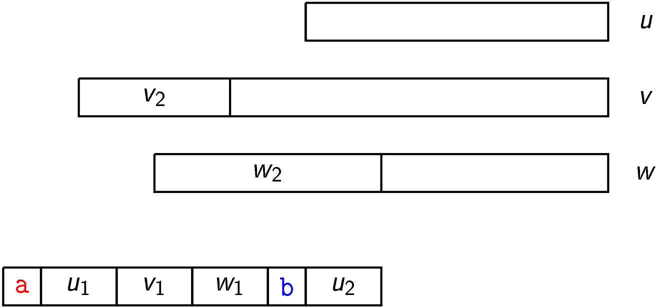

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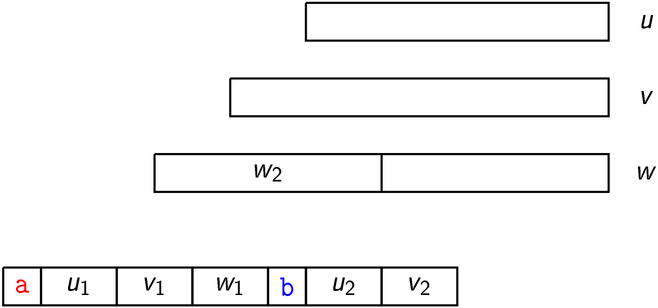

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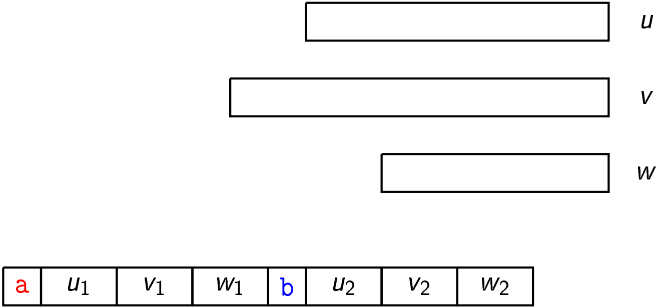

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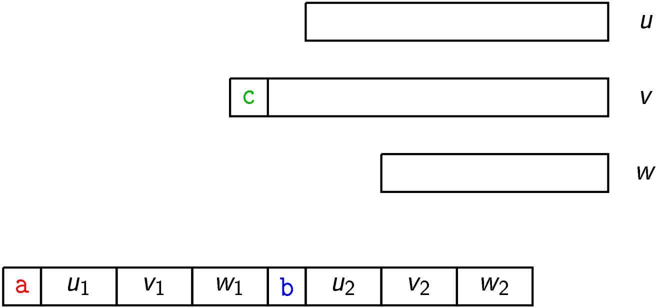

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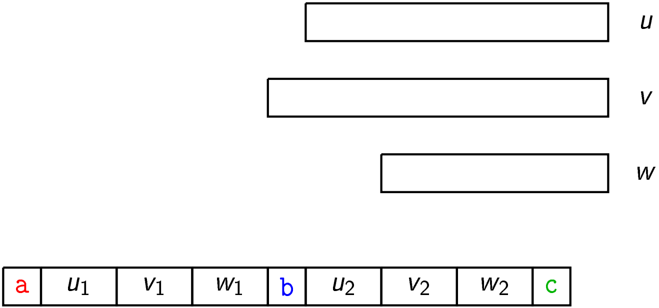

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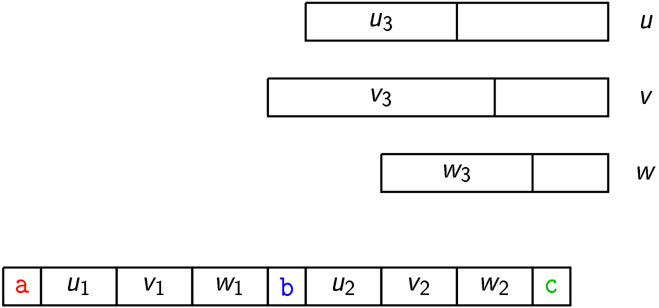

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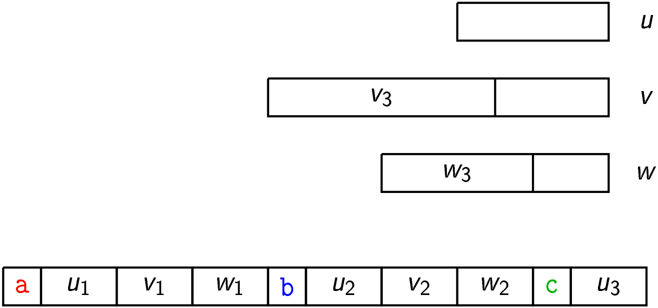

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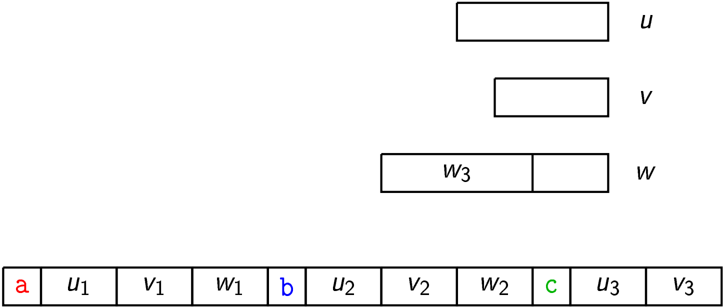

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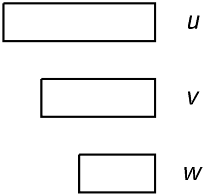

a	u_1	v_1	w_1	b	u_2	v_2	w_2	c	u_3	v_3	w_3

End

Thank you for your attention.